Equivalence between transfer-matrix and observed-state feedback control

C.R. Rojas, R.A. Rojas and M.E. Salgado

Abstract: An observed-state feedback is built for a given multiple input–multiple output (MIMO) control loop, where the controller is specified in transfer-matrix form. This contribution solves for the first time, for MIMO systems, the classical problem of finding a feedback gain and an observer gain such that the observed-state feedback control loop has the same sensitivity as that provided by a one-degree-of-freedom classical control loop.

1 Introduction

One of the pillars of modern linear control theory is the observed-state feedback of a system described in state-space form. This strategy has been thoroughly studied since state-space representation can provide information on the system properties which is not present in the transfer-matrix description. One of the advantages of the control design based on state-space techniques lies on the decoupling of some of the major control issues, such as regulation, reference tracking, state estimation and disturbance rejection. In particular, observed-state feedback simplifies the control synthesis via the separation principle.

On the other hand, classical control theory, with its input–output description, is closer to control engineering and has some advantages due to its treatment of control loop sensitivity and robustness against modelling errors.

However, since modern and classic approaches can be used to solve the same linear control problem, an understanding of their connection should provide useful insight. In other words, unification of both approaches to control contributes to a more complete vision of design and its specifications and, among other benefits, can help to understand the degrees-of-freedom available to the control designer. In particular, since design is usually an iterative process, a procedure to establish an explicit equivalence between classical and modern approaches to control may be useful to study and to improve the time-domain and frequency-domain properties of a particular controller design, by taking advantage of both alternatives.

It is important to realise that, although both theories can be used to solve the same control problems, it is not clear that they yield the same solutions (in an input–output sense). This is because it is not known whether the controller structures given by classical control and observed-state control theories provide exactly the same degrees-of-freedom.

Equivalence between classical and modern control strategies has been only recently studied by Yuz and Salgado [1]. They proved that, for scalar plants, every proper controller is equivalent to an observed-state feedback controller. To the best knowledge of the authors, the problem has not yet been solved for the multivariable case. Moreover, and as discussed in the following section of the paper, the approach in [1] cannot be extended to multivariable plants. This article advances the solution to the equivalence problem for multiple input–multiple output (MIMO) systems.

This paper contributes to the unification of classical and modern control theories in a linear multivariable framework. Here, we consider a multivariable plant under one-degree-of-freedom feedback control, and the same plant under observed-state feedback control. The problem we deal with is: given the first control architecture, with a controller in transfer-matrix form, find the feedback gain and the observer gain, for the second architecture, so that both control loops exhibit the same sensitivity transfer-matrix. In this paper, we solve the problem for a square plant with a strictly proper controller of any order.

2 The equivalence problem

2.1 Problem statement

Consider a plant having a minimal realisation given by

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t)\end{aligned} \tag{1}$$

where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B} \in \mathbb{R}^{n\times m}$, $\boldsymbol{C} \in \mathbb{R}^{p\times n}$; $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state of the system, $\boldsymbol{u}(t) \in \mathbb{R}^m$ is the input and $\boldsymbol{y}(t) \in \mathbb{R}^p$ is the output. In the sequel we assume that $m = p$.

Assume that the plant should follow a given reference $\boldsymbol{r}(t) \in \mathbb{R}^p$ and that the chosen architecture is the feedback of the observed-state. The observer equation and the feedback control law are given by

$$\begin{aligned}\dot{\hat{\boldsymbol{x}}}(t) &= \boldsymbol{A}\hat{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{J}(\boldsymbol{y}(t) - \boldsymbol{C}\hat{\boldsymbol{x}}(t))\\ &= (\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})\hat{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{J}\boldsymbol{y}(t)\\ \boldsymbol{u}(t) &= -\boldsymbol{K}\hat{\boldsymbol{x}}(t) + \bar{\boldsymbol{r}}(t)\end{aligned} \tag{2}$$

where $\bar{\boldsymbol{r}}(t) \in \mathbb{R}^m$ is a filtered reference signal, $\hat{\boldsymbol{x}}(t) \in \mathbb{R}^n$ is the observer state and $\boldsymbol{J} \in \mathbb{R}^{n\times p}$ is the observer gain.

IEE Proceedings online no. 20050234
doi:10.1049/ip-cta:20050234
Paper first received 10th January 2005

The authors are with the Department of Electronic Engineering, Universidad Técnica Federico Santa María, Valparaíso, Chile
E-mail: cristian.rojase@gmail.com

Then, it can be shown that (see, e.g. [2]):

$$U(s) = [I + K(sI - A + JC)^{-1}B]^{-1} \times [-K(sI - A + JC)^{-1}JY(s) + \bar{R}(s)] \quad (3)$$

If the filtered reference satisfies

$$\bar{R}(s) = K(sI - A + JC)^{-1}JR(s) = \left[\begin{array}{cc} A - JC & J \\ K & 0 \end{array}\right] R(s) \quad (4)$$

where the notation $\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]$ stands for $C(sI - A)^{-1}B + D$, then

$$U(s) = C(s)(R(s) - Y(s)) \quad (5)$$

where

$$C(s) = K(sI - A + JC + BK)^{-1}J \quad (6)$$

with $U(s) = \mathcal{L}\{u(t)\}$, $R(s) = \mathcal{L}\{r(t)\}$, $\bar{R}(s) = \mathcal{L}\{\bar{r}(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$.

The above equations show that given the plant state model (A, B, C, 0), the feedback gain K, and the full-order observer gain J, there is always an equivalent classical controller, which is strictly proper and having a degree equal to the plant degree.

In this paper, we investigate the inverse equivalence problem, that is, given $C(s)$, strictly proper and having the same degree as the plant, find K and J. We will later relax the degree requirement.

In the sequel, to make the notion of equivalence more precise, we will say that two systems are externally equivalent if their transfer matrices coincide, and two systems will be said to be internally equivalent if their state-space realisations can be related by a similarity transformation.

2.2 Equivalence for a scalar loop

Yuz and Salgado [1] proved, for the single input–single output (SISO) case, that every classical, internally stable, control loop is externally equivalent to a observed-state feedback control loop. That result includes a procedure to find K and J, given the controller transfer function $C(s)$ having the same degree to that of the plant.

The key concept underlying the results in [1] is the fact that the controller $C(s)$ uniquely determines (modulo stable cancellations) the closed-loop characteristic polynomial $A_{cl}(s)$, and that $A_{cl}(s) = \det(sI - A + BK) \cdot \det(sI - A + JC)$. Therefore although there might be an infinite number of pairs (J, K) leading to the same $A_{cl}(s)$, there is a unique SISO controller $C(s)$.

The above key idea does not hold in the MIMO case, since the closed-loop characteristic polynomial $A_{cl}(s)$ is not uniquely tied to a given MIMO controller. For more details, see chapter 7 of [9].

3 Equivalence based on a Riccati equation

3.1 Approach description

According to Luenberger, almost any system can be seen as an observer of another one (see Note at the end of this section, and [4–6]). This idea may be used to consider a transfer-matrix controller of same order as the plant as a combination of a state-feedback and a full-order observer. To this end, consider again system (1) and a controller with realisation

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c[r(t) - y(t)] \\ u(t) &= C_c x_c(t) \end{aligned} \quad (7)$$

where $x_c(t) \in \mathbb{R}^n$ is the controller state, $A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times p}$ and $C_c \in \mathbb{R}^{m\times n}$.

The idea of Luenberger lies on the fact that, under a suitable state transformation, the controller state is an estimation of the actual plant state. To see that, we need to modify (7) so that the controller state $x_c(t)$ depends only on $u(t)$ and $y(t)$, since an observer is driven by the input and the output of the observed system.

To include the plant input $u(t)$ into the controller state equation, we add and subtract $\Gamma u(t)$ (where $\Gamma \in \mathbb{R}^{n\times m}$ remains to be determined) in the first equation of (7), so we obtain

$$\begin{aligned} \dot{x}_c(t) &= (A_c - \Gamma C_c)x_c(t) + \Gamma u(t) + B_c r(t) - B_c y(t) \\ u(t) &= C_c x_c(t) \end{aligned} \quad (8)$$

To simplify these expressions, we define

$$\tilde{A}_c = A_c - \Gamma C_c \quad (9)$$

Then, to eliminate $r(t)$ from (8), we use the linearity of the controller to express its state as a sum of two vectors, $x_o(t)$ and $x_R(t)$, where $x_o(t)$ depends only on $u(t)$ and $y(t)$, and $x_R(t)$ depends exclusively on $r(t)$. In this way, we obtain

$$\begin{aligned} \dot{x}_o(t) &= \tilde{A}_c x_o(t) + \Gamma u(t) - B_c y(t) \\ \dot{x}_R(t) &= \tilde{A}_c x_R(t) + B_c r(t) \\ u(t) &= C_c x_o(t) + C_c x_R(t) \end{aligned} \quad (10)$$

Therefore $x_o(t)$ may be interpreted as the state of an observer, and $x_R(t)$ may be considered as the state of a reference prefilter. To highlight the relationship between (10) and the equations of a state-space controller, we define $\bar{r}(t) = C_c x_R(t)$, so (10) may be written as

$$\begin{aligned} \dot{x}_o(t) &= \tilde{A}_c x_o(t) + \Gamma u(t) - B_c y(t) \\ \dot{x}_R(t) &= \tilde{A}_c x_R(t) + B_c r(t) \\ \bar{r}(t) &= C_c x_R(t) \\ u(t) &= C_c x_o(t) + \bar{r}(t) \end{aligned} \quad (11)$$

Suppose, then, that $x_o(t) = Tx(t)$, where $T \in \mathbb{R}^{n\times n}$ is a suitable (non-singular) transformation matrix. Using (1) and (11), we have

$$\begin{aligned} T\dot{x}(t) &= TAx(t) + TBu(t) \\ T\dot{x}(t) &= \tilde{A}_c Tx(t) - B_c Cx(t) + \Gamma u(t) \end{aligned} \quad (12)$$

Since the above equations hold for every t and every $u(t)$ and $x(t)$, we have that

$$\begin{aligned} \Gamma &= TB \\ TA - \tilde{A}_c T &= -B_c C \end{aligned} \quad (13)$$

Next, from (9) it follows that

$$\tilde{A}_c = A_c - TBC_c \quad (14)$$

Introducing (14) into (13), we get

$$TA - (A_c - TBC_c)T = -B_c C \quad (15)$$

This means that the transformation matrix $\boldsymbol{T}$ must be an invertible solution of the Riccati equation

$$\boldsymbol{TA} - \boldsymbol{A}_c\boldsymbol{T} + \boldsymbol{TBC}_c\boldsymbol{T} = -\boldsymbol{B}_c\boldsymbol{C} \tag{16}$$

The previous arguments prove that an internal equivalence implies the existence of an invertible solution of (16). Now we prove the converse, that is, that every invertible solution of (16) gives an internal equivalence. To do this, let $\boldsymbol{T}$ be an invertible solution of (16), so we can define the transformed state

$$\begin{aligned}\bar{\boldsymbol{x}}_o(t) &= \boldsymbol{T}^{-1}\boldsymbol{x}_o(t)\\ \bar{\boldsymbol{x}}_R(t) &= \boldsymbol{T}^{-1}\boldsymbol{x}_R(t)\end{aligned} \tag{17}$$

Under this transformation, the controller is given by

$$\begin{aligned}\dot{\bar{\boldsymbol{x}}}_o(t) &= \boldsymbol{T}^{-1}(\boldsymbol{A}_c - \boldsymbol{TBC}_c)\boldsymbol{T}\bar{\boldsymbol{x}}_o(t)\\ &\quad + \boldsymbol{Bu}(t) - \boldsymbol{T}^{-1}\boldsymbol{B}_c\boldsymbol{y}(t)\\ \dot{\bar{\boldsymbol{x}}}_R(t) &= \boldsymbol{T}^{-1}(\boldsymbol{A}_c - \boldsymbol{TBC}_c)\boldsymbol{T}\bar{\boldsymbol{x}}_R(t) + \boldsymbol{T}^{-1}\boldsymbol{B}_c\boldsymbol{r}(t)\\ \bar{\boldsymbol{r}}(t) &= \boldsymbol{C}_c\boldsymbol{T}\bar{\boldsymbol{x}}_R(t)\\ \boldsymbol{u}(t) &= \boldsymbol{C}_c\boldsymbol{T}\bar{\boldsymbol{x}}_o(t) + \bar{\boldsymbol{r}}(t)\end{aligned} \tag{18}$$

Therefore it is true that

$$\begin{aligned}\dot{\boldsymbol{x}}(t) - \dot{\bar{\boldsymbol{x}}}_o(t) &= \boldsymbol{Ax}(t) - \boldsymbol{T}^{-1}\boldsymbol{A}_c\boldsymbol{T}\bar{\boldsymbol{x}}_o(t) + \boldsymbol{BC}_c\boldsymbol{T}\bar{\boldsymbol{x}}_o(t)\\ &\quad + \boldsymbol{T}^{-1}\boldsymbol{B}_c\boldsymbol{Cx}(t)\\ &= [\boldsymbol{A} + \boldsymbol{T}^{-1}\boldsymbol{B}_c\boldsymbol{C}]\boldsymbol{x}(t)\\ &\quad + [\boldsymbol{BC}_c\boldsymbol{T} - \boldsymbol{T}^{-1}\boldsymbol{A}_c\boldsymbol{T}]\bar{\boldsymbol{x}}_o(t)\end{aligned} \tag{19}$$

However, from (16) it follows that

$$\begin{aligned}\boldsymbol{TBC}_c\boldsymbol{T} - \boldsymbol{A}_c\boldsymbol{T} &= -\boldsymbol{B}_c\boldsymbol{C} - \boldsymbol{TA}\\ &\Longrightarrow \boldsymbol{BC}_c\boldsymbol{T} - \boldsymbol{T}^{-1}\boldsymbol{A}_c\boldsymbol{T}\\ &= -\boldsymbol{T}^{-1}\boldsymbol{B}_c\boldsymbol{C} - \boldsymbol{A}\end{aligned} \tag{20}$$

so

$$\dot{\boldsymbol{x}}(t) - \dot{\bar{\boldsymbol{x}}}_o(t) = [\boldsymbol{A} + \boldsymbol{T}^{-1}\boldsymbol{B}_c\boldsymbol{C}](\boldsymbol{x}(t) - \bar{\boldsymbol{x}}_o(t)) \tag{21}$$

This implies that $\bar{\boldsymbol{x}}_o(t)$ estimates the state of system (1). Henceforth, using (20), (18) may be written as

$$\begin{aligned}\dot{\bar{\boldsymbol{x}}}_o(t) &= (\boldsymbol{A} + \boldsymbol{T}^{-1}\boldsymbol{B}_c\boldsymbol{C})\bar{\boldsymbol{x}}_o(t) + \boldsymbol{Bu}(t) - \boldsymbol{T}^{-1}\boldsymbol{B}_c\boldsymbol{y}(t)\\ \dot{\bar{\boldsymbol{x}}}_R(t) &= (\boldsymbol{A} + \boldsymbol{T}^{-1}\boldsymbol{B}_c\boldsymbol{C})\bar{\boldsymbol{x}}_R(t) + \boldsymbol{T}^{-1}\boldsymbol{B}_c\boldsymbol{r}(t)\\ \bar{\boldsymbol{r}}(t) &= \boldsymbol{C}_c\boldsymbol{T}\bar{\boldsymbol{x}}_R(t)\\ \boldsymbol{u}(t) &= \boldsymbol{C}_c\boldsymbol{T}\bar{\boldsymbol{x}}_o(t) + \bar{\boldsymbol{r}}(t)\end{aligned} \tag{22}$$

Rearranging the terms, we obtain

$$\begin{aligned}\dot{\bar{\boldsymbol{x}}}_o(t) &= \boldsymbol{A}\bar{\boldsymbol{x}}_o(t) + \boldsymbol{Bu}(t) - \boldsymbol{T}^{-1}\boldsymbol{B}_c[\boldsymbol{y}(t) - \boldsymbol{C}\bar{\boldsymbol{x}}_o(t)]\\ \dot{\bar{\boldsymbol{x}}}_R(t) &= (\boldsymbol{A} + \boldsymbol{T}^{-1}\boldsymbol{B}_c\boldsymbol{C})\bar{\boldsymbol{x}}_R(t) + \boldsymbol{T}^{-1}\boldsymbol{B}_c\boldsymbol{r}(t)\\ \bar{\boldsymbol{r}}(t) &= \boldsymbol{C}_c\boldsymbol{T}\bar{\boldsymbol{x}}_R(t)\\ \boldsymbol{u}(t) &= \boldsymbol{C}_c\boldsymbol{T}\bar{\boldsymbol{x}}_o(t) + \bar{\boldsymbol{r}}(t)\end{aligned} \tag{23}$$

Comparing the above expressions with those of a state-space controller

$$\begin{aligned}\dot{\bar{\boldsymbol{x}}}_o(t) &= \boldsymbol{A}\bar{\boldsymbol{x}}_o(t) + \boldsymbol{Bu}(t) + \boldsymbol{J}[\boldsymbol{y}(t) - \boldsymbol{C}\bar{\boldsymbol{x}}_o(t)]\\ \dot{\bar{\boldsymbol{x}}}_R(t) &= (\boldsymbol{A} - \boldsymbol{JC})\bar{\boldsymbol{x}}_R(t) + \boldsymbol{Jr}(t)\\ \bar{\boldsymbol{r}}(t) &= \boldsymbol{K}\bar{\boldsymbol{x}}_R(t)\\ \boldsymbol{u}(t) &= -\boldsymbol{K}\bar{\boldsymbol{x}}_o(t) + \bar{\boldsymbol{r}}(t)\end{aligned} \tag{24}$$

it follows that, after changing the sign of the reference pre-filter state, $\boldsymbol{x}_R(t)$, (7) can be seen as a state-space controller with gain matrices

$$\begin{aligned}\boldsymbol{J} &= -\boldsymbol{T}^{-1}\boldsymbol{B}_c\\ \boldsymbol{K} &= -\boldsymbol{C}_c\boldsymbol{T}\end{aligned} \tag{25}$$

We have thus shown that the problem of existence and uniqueness of an internal equivalence between a strictly proper multivariable controller and a state-space controller given by a full-order observer and a state-feedback is analogous to the existence and uniqueness of invertible solutions of the Riccati equation (16).

In short, the resulting procedure to obtain matrices $\boldsymbol{J}$ and $\boldsymbol{K}$ from (7) is as follows:

Procedure.

1. Describe the plant and controller in state-space form, according to (1) and (7).
2. Solve Riccati equation (16).
3. The controller can be described as in (24), where $\boldsymbol{J}$ and $\boldsymbol{K}$ are given by (25).

Note. The term 'almost' used by Luenberger when he said that 'almost' any system can be seen as an observer of another one comes from the fact that Luenberger [4] showed this for the case in which both systems have no common eigenvalues. If this is not so, it may happen that the first system cannot be seen as an observer of the second one, or that there may exist infinite ways to establish this relationship between the systems. The condition of Luenberger establishes the existence and uniqueness of solutions for the second equation of (13) (in $\boldsymbol{T}$) if $\boldsymbol{A}$ and $\tilde{\boldsymbol{A}}_c$ are independent of $\boldsymbol{T}$; however, $\tilde{\boldsymbol{A}}_c$ does depend on $\boldsymbol{T}$, so his condition is not applicable here.

3.2 Solving the Riccati equation

Theorem 1 in the Appendix says that the invertible solutions of (16) have the form $\boldsymbol{T} = \boldsymbol{GF}^{-1}$, where columns of matrix $[\boldsymbol{F}^T \quad \boldsymbol{G}^T]^T$ are generalised right eigenvectors of the pseudo-Hamiltonian matrix.

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{BC}_c \\ -\boldsymbol{B}_c\boldsymbol{C} & \boldsymbol{A}_c \end{bmatrix} \tag{26}$$

[This matrix is built from the asymmetric Riccati equation in the same way as the Hamiltonian matrix of a symmetric Riccati equation, although strictly speaking $\boldsymbol{H}$ is not a Hamiltonian matrix, since its eigenvalues do not satisfy any kind of symmetry.]

Generalised right eigenvectors of $\boldsymbol{H}$ chosen to create $[\boldsymbol{F}^T \quad \boldsymbol{G}^T]^T$ must satisfy the three conditions of Theorem 1 in the Appendix, and $\boldsymbol{G}$ must be invertible.

The discussion above shows that the problem of finding an internal equivalence between a given controller and the combination of an observer and a state-feedback, can be reduced to the problem of finding n right generalised eigenvectors of $\boldsymbol{H}$ satisfying certain properties. Using an exhaustive search, we would have to try, in the worst case (for a plant with distinct real eigenvalues)

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \tag{27}$$

possibilities.

For $n = 5$, for example, 252 cases should be analysed. To reduce the number of computations, necessary and sufficient conditions on the existence of real invertible solutions of (16) would be welcome.

Note. In general it is unnecessary to know the generalised eigenvectors of $\boldsymbol{H}$ to solve Riccati equation (16), because it is enough to choose a set of linearly independent column vectors whose span coincides with the column space of $[\boldsymbol{F}^T \quad \boldsymbol{G}^T]^T$ [7].

3.3 Interpretation

By combining the equations defining the plant, (1), and the controller, (7), we obtain the closed-loop equations

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{C}_c\boldsymbol{x}_c(t)\\ \dot{\boldsymbol{x}}_c(t) &= \boldsymbol{A}_c\boldsymbol{x}_c(t) - \boldsymbol{B}_c\boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{B}_c\boldsymbol{r}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t)\end{aligned} \tag{28}$$

which may be written in matrix form as

$$\begin{bmatrix}\dot{\boldsymbol{x}}(t)\\ \dot{\boldsymbol{x}}_c(t)\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{B}\boldsymbol{C}_c\\ -\boldsymbol{B}_c\boldsymbol{C} & \boldsymbol{A}_c\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{x}_c(t)\end{bmatrix} + \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{B}_c\end{bmatrix}\boldsymbol{r}(t)$$

$$\boldsymbol{y}(t) = \begin{bmatrix}\boldsymbol{C} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{x}_c(t)\end{bmatrix} \tag{29}$$

where we see that the closed-loop state matrix is precisely the pseudo-Hamiltonian matrix (26).

By applying the invertible state transformation matrix

$$\boldsymbol{U} = \begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{T} & -\boldsymbol{T}\end{bmatrix} \tag{30}$$

together with expressions (16), (20) and (25), one observes that the closed-loop can be described as

$$\begin{bmatrix}\dot{\boldsymbol{x}}'(t)\\ \dot{\boldsymbol{x}}_c'(t)\end{bmatrix} = \begin{bmatrix}\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K} & \boldsymbol{B}\boldsymbol{K}\\ \boldsymbol{0} & \boldsymbol{A}-\boldsymbol{J}\boldsymbol{C}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}'(t)\\ \boldsymbol{x}_c'(t)\end{bmatrix} + \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{J}\end{bmatrix}\boldsymbol{r}(t)$$

$$\boldsymbol{y}(t) = \begin{bmatrix}\boldsymbol{C} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}'(t)\\ \boldsymbol{x}_c'(t)\end{bmatrix} \tag{31}$$

Therefore matrix $\boldsymbol{T}$ allows to transform the closed-loop state to that of a combination of a full-order observer and a state-feedback.

Also, from (25) we know that $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K} = \boldsymbol{A} + \boldsymbol{B}\boldsymbol{C}_c\boldsymbol{T}$, so from Theorem 1 we conclude that $\sigma(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})$ is the set of eigenvalues of $\boldsymbol{H}$ associated with the generalised right eigenvectors chosen to build $\boldsymbol{T}$. [$\sigma(\boldsymbol{A})$ is the spectrum of $\boldsymbol{A}$, i.e. the set of its eigenvalues.]

Note 1. An alternative way to establish the internal equivalence between the control structures, also leading to (16), is to observe that what we need is to express the controller, given by (7), in the form

$$\begin{aligned}\boldsymbol{C}(s) &= \left[\begin{array}{c|c}\boldsymbol{A}-\boldsymbol{J}\boldsymbol{C}-\boldsymbol{B}\boldsymbol{K} & \boldsymbol{J}\\ \hline \boldsymbol{K} & \boldsymbol{0}\end{array}\right]\\ &= \boldsymbol{K}(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{J}\boldsymbol{C} + \boldsymbol{B}\boldsymbol{K})^{-1}\boldsymbol{J}\end{aligned} \tag{32}$$

using an invertible state transformation matrix $\boldsymbol{T} \in \mathbb{R}^{n\times n}$. This matrix must satisfy the equations

$$\begin{aligned}\boldsymbol{T}^{-1}\boldsymbol{A}_c\boldsymbol{T} &= \boldsymbol{A} - \boldsymbol{J}\boldsymbol{C} - \boldsymbol{B}\boldsymbol{K}\\ \boldsymbol{T}^{-1}\boldsymbol{B}_c &= \boldsymbol{J}\\ \boldsymbol{C}_c\boldsymbol{T} &= \boldsymbol{K}\end{aligned} \tag{33}$$

By replacing the second and the third equations into the first one, and pre-multiplying by $\boldsymbol{T}$, we obtain the Riccati equation

$$\boldsymbol{A}_c\boldsymbol{T} = \boldsymbol{T}\boldsymbol{A} - \boldsymbol{B}_c\boldsymbol{C} - \boldsymbol{T}\boldsymbol{B}\boldsymbol{C}_c\boldsymbol{T} \tag{34}$$

which corresponds essentially to (16), after changing $\boldsymbol{T}$ for $-\boldsymbol{T}$. This sign discrepancy disappears if we change $\boldsymbol{x}_o(t)$ for $-\boldsymbol{x}_o(t)$ in (10).

Note 2. The results derived above are valid for continuous-time as well as for discrete-time systems, because state representations for both kinds of systems have essentially the same structure. Henceforth, the Riccati equation (16) is equally valid to represent discrete-time controllers as a combination of a state-feedback and a full-order observer. This is neither intuitive nor trivial, because (16) seems to be an extension of what is known as a continuous-time (symmetric) Riccati equation. Thus, by analogy, we may expect to find that for discrete-time systems we would need an extension of a discrete-time (symmetric) Riccati equation. However, our interpretation to (16) is very different to that usually given to continuous-time symmetric Riccati equations (see, for instance, [3]), so the analogy between symmetric and asymmetric Riccati equations does not apply here.

Note 3. The fact that the eigenvalues associated with the generalised right eigenvectors used to build $\boldsymbol{T}$ are the eigenvalues of $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$ seems to violate the existing symmetry between the expressions of a state-feedback and a full-order observer. However, such asymmetry comes from using generalised 'right' eigenvectors. It can be shown [8] that if we use generalised left eigenvectors of $\boldsymbol{H}$ to build solutions of the Riccati equation (16), the eigenvalues associated with these eigenvectors are those of $\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C}$.

Example 1. Equivalence with a Riccati equation: Consider a plant whose state-variable representation has matrices

$$\boldsymbol{A} = \begin{bmatrix}-1 & -2\\ 0 & 1\end{bmatrix}; \quad \boldsymbol{B} = \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}; \quad \boldsymbol{C} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \tag{35}$$

This plant has poles at $\{-1, 1\}$. If we use a state-feedback matrix given by

$$\boldsymbol{K} = \begin{bmatrix}-2/3 & 4\\ 4/3 & -3\end{bmatrix} \tag{36}$$

and an observer with gain

$$\boldsymbol{J} = \begin{bmatrix}6 & -2\\ -7 & 7\end{bmatrix} \tag{37}$$

Then, we see that the controller transfer-matrix is

$$\boldsymbol{C}(s) = \begin{bmatrix}\dfrac{-32(s+7.208)}{(s+11)(s+7)} & \dfrac{29.33(s+6.864)}{(s+11)(s+7)}\\[2ex] \dfrac{29(s+7.460)}{(s+11)(s+7)} & \dfrac{-23.67(s+6.662)}{(s+11)(s+7)}\end{bmatrix} \tag{38}$$

For this transfer-matrix, an alternative state representation is

$$\boldsymbol{C}(s) = \left[\begin{array}{cc|cc}-7 & 0 & 2.5 & 1.5\\ 0 & -11 & -7.826 & 7.826\\ \hline -0.6667 & 3.876 & 0 & 0\\ 1.333 & -3.280 & 0 & 0\end{array}\right] \tag{39}$$

For this realisation, Riccati equation (16) has the form

$$\begin{aligned}&\boldsymbol{T}\begin{bmatrix}-1 & -2\\ 0 & 1\end{bmatrix} - \begin{bmatrix}-7 & 0\\ 0 & -11\end{bmatrix}\boldsymbol{T} + \boldsymbol{T}\begin{bmatrix}2 & -2.683\\ 0 & 4.472\end{bmatrix}\boldsymbol{T}\\ &\quad = \begin{bmatrix}-4 & -1.5\\ 0 & -7.826\end{bmatrix}\end{aligned} \tag{40}$$

The pseudo-Hamiltonian matrix associated with this equation is

$$\boldsymbol{H} = \begin{bmatrix} -1 & -2 & 2 & -2.683 \\ 0 & 1 & 0 & 4.472 \\ -4 & -1.5 & -7 & 0 \\ 0 & -7.826 & 0 & -11 \end{bmatrix} \tag{41}$$

whose eigenvalues are $\lambda_1 = -3$, $\lambda_2 = -4$, $\lambda_3 = -5$ and $\lambda_4 = -6$. Associated right eigenvectors are respectively

$$\boldsymbol{v}_1 = \begin{bmatrix} -1.417 \\ 0 \\ 1.417 \\ 0 \end{bmatrix}; \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 3.5 \\ -1.75 \\ -3.913 \end{bmatrix}$$

$$\boldsymbol{v}_3 = \begin{bmatrix} 0.375 \\ 0 \\ -0.75 \\ 0 \end{bmatrix}; \quad \boldsymbol{v}_4 = \begin{bmatrix} 0.6667 \\ -2.5 \\ 1.083 \\ 3.913 \end{bmatrix} \tag{42}$$

Equation (40) has multiple solutions, but each one of them can be uniquely characterised by a two-element set $S \subset \sigma(\boldsymbol{H})$.

If we choose $S = \{-3, -4\}$, which is consistent with the initial selection of $\boldsymbol{K}$ matrix in (36), the solution of equation (40) is

$$\boldsymbol{T} = \begin{bmatrix} -1 & -0.5 \\ 0 & -1.118 \end{bmatrix} \tag{43}$$

so from (25) we obtain

$$\boldsymbol{J} = \begin{bmatrix} 6 & -2 \\ -7 & 7 \end{bmatrix}; \quad \boldsymbol{K} = \begin{bmatrix} -0.6667 & 4 \\ 1.333 & -3 \end{bmatrix} \tag{44}$$

These matrices coincide with the observer and state-feedback gain matrices ((37) and (36), respectively).

For $S = \{-3, -5\}$ there is no solution, because the upper and lower second-order submatrices of $[\boldsymbol{v}_1 \quad \boldsymbol{v}_3]$ are singular. In all, there are five equivalences for this problem [7].

As we can see from this last example, and unlike what happens with SISO loops, not every choice of S gives an internal equivalence in MIMO loops; there are now cases where the number of degrees-of-freedom available for equivalence is less than expected at first sight.

3.4 Necessary condition for equivalence

A necessary condition for a given matrix $\boldsymbol{T} = \boldsymbol{G}\boldsymbol{F}^{-1}$ to be a real invertible solution of (16) is given by Theorem 2. To state this condition, it is convenient to define S as the set of eigenvalues associated with generalised right eigenvectors chosen to form $[\boldsymbol{F}^T \quad \boldsymbol{G}^T]^T$, and to define $\bar{S}$ as the remaining eigenvalues of $\boldsymbol{H}$ (even if some of them coincide with those of S). Then, for $\boldsymbol{T}$ to be a real invertible solution of (16) it is necessary that all uncontrollable eigenvalues of pairs $(\boldsymbol{A}, \boldsymbol{B}\boldsymbol{C}_c)$ and $(\boldsymbol{A}_c, -\boldsymbol{B}_c\boldsymbol{C})$ belong to S, and that all unobservable eigenvalues of pairs $(\boldsymbol{A}, -\boldsymbol{B}_c\boldsymbol{C})$ and $(\boldsymbol{A}_c, \boldsymbol{B}\boldsymbol{C}_c)$ belong to $\bar{S}$. [According to the PBH tests (see, e.g. [9]), the pair $(\boldsymbol{A}, \boldsymbol{B})$ $((\boldsymbol{A}, \boldsymbol{C}))$ is controllable (observable) if and only if there is no vector $\boldsymbol{v}$ such that $\boldsymbol{v}^T\boldsymbol{A} = \lambda\boldsymbol{v}^T$ $(\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v})$ and $\boldsymbol{v}^T\boldsymbol{B} = 0$ $(\boldsymbol{C}\boldsymbol{v} = 0)$ for some $\lambda \in \mathbb{C}$. A λ for which there is such a vector is called an uncontrollable (unobservable) eigenvalue of $(\boldsymbol{A}, \boldsymbol{B})$ $((\boldsymbol{A}, \boldsymbol{C}))$; the remaining eigenvalues of $\boldsymbol{A}$ are controllable (observable).]

The previous condition may be simplified by using Theorem 3, and assuming that columns of $\boldsymbol{B}$ and $\boldsymbol{B}_c$ are linearly independent, as also are rows of $\boldsymbol{C}$ and $\boldsymbol{C}_c$, and so the condition may be stated as follows: For $\boldsymbol{T}$ to be a real invertible solution of (16) it is necessary that all uncontrollable eigenvalues of pairs $(\boldsymbol{A}, \boldsymbol{B})$ and $(\boldsymbol{A}_c, \boldsymbol{B}_c)$ belong to S, and that all unobservable eigenvalues of pairs $(\boldsymbol{A}, \boldsymbol{C})$ and $(\boldsymbol{A}_c, \boldsymbol{C}_c)$ belong to $\bar{S}$. This means that all uncontrollable eigenvalues of the plant and the controller must be in S, and that all their unobservable eigenvalues must belong to $\bar{S}$.

3.4.1 Infinite number of solutions of a Riccati equation:

As already stated in Section 3.2, the number of solutions of a Riccati equation is finite when its pseudo-Hamiltonian matrix $\boldsymbol{H}$ has distinct eigenvalues. However, if $\boldsymbol{H}$ is derogatory (see, e.g. [8]), that is, if it has two or more linearly independent eigenvectors associated with the same eigenvalue, then the Riccati equation may have an infinite number of solutions (although not necessarily invertible ones). To better understand this property, assume, for instance, that $\boldsymbol{H}$ has at least two linearly independent eigenvectors, say $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, associated with the same eigenvalue in λ_1. Hence, if we choose the set S so that it includes exactly one of these eigenvalues, then there exists an infinite number of possible eigenvectors related to λ_1 which may be used to obtain a solution to the Riccati equation, because every linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ is an eigenvector of $\boldsymbol{H}$ associated with λ_1; each one of such eigenvectors may give a different solution, so the Riccati equation may have an infinite number of solutions.

Even if the Riccati equation (16) may have, in principle, an infinite number of solutions, it is not necessarily true that there exists an infinite number of matrices $\boldsymbol{J}$ and $\boldsymbol{K}$ leading to the same controller $\boldsymbol{C}(s)$. For instance, in the case of a scalar loop, as $\boldsymbol{H}$ is the closed-loop state matrix, the fact of it being derogatory implies that the loop has at least one uncontrollable and unobservable mode [7]; this mode cannot come from the plant (since it has been assumed that it is given by a minimal realisation), so it must come from the controller or from pole-zero cancellations between the controller and the plant. Also, according to Section 3.2, a strictly proper scalar controller given by a transfer function is always externally equivalent to the combination of a state-feedback and a full-order observer, and there is a bijective relation between the gain matrices of these systems and the closed-loop eigenvalues (split into two sets of the same size). Hence, if the controller is internally equivalent to one of those combinations, matrices $\boldsymbol{J}$ and $\boldsymbol{K}$ are unique.

The reasoning above does not hold in the MIMO case, since the fact that the closed-loop state matrix is derogatory does not imply that the loop has uncontrollable and/or unobservable modes (see, for instance, [9]), and the closed-loop eigenvalues do not determine uniquely the gain matrices of the state-feedback and the observer. However, to the best of our knowledge, there are no cases where an infinite number of matrices $\boldsymbol{J}$ and $\boldsymbol{K}$ leads to the equivalence between a classical controller and a state-space controller.

The analysis of the case when a loop yields a Riccati equation with infinite invertible solutions, can be found in [7].

Sometimes a Riccati equation may have infinite solutions, but none of them are invertible; see, for instance, [10].

3.5 Cancellations between the controller and the plant

The theory developed in the previous sections is still valid when there are cancellations between poles and zeros of the plant with zeros and poles of the controller, respectively. For illustrative examples see [7].

4 Controllers of lower degree than the plant

In this section, we extend the method developed previously, to establish the external equivalence between a strictly proper controller of lower order than the plant and the combination of a state-feedback and a full-order observer. To do this, we use the Kalman decomposition theorem to introduce uncontrollable and/or unobservable modes into the controller. In this way we build a controller whose realisation has the same order as the plant, so we can proceed as before.

Consider a controller given by a transfer-matrix, of lower order than the plant. It is possible to represent this controller by (7), with $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times p}$, $C_c \in \mathbb{R}^{m \times n_c}$ and $n_c < n$.

The method previously developed is valid only when $n_c = n$. Hence, it is necessary to extend the controller model to an nth order realisation by adding stable uncontrollable and/or unobservable subsystems into the controller. This can be done with the Kalman decomposition theorem, leading (7) to

$$\begin{bmatrix} \dot{x}_c(t) \\ \dot{x}_{c,\bar{o}}(t) \\ \dot{x}_{\bar{c},o}(t) \\ \dot{x}_{\bar{c},\bar{o}}(t) \end{bmatrix} = \begin{bmatrix} A_c & 0 & A_{1,3} & 0 \\ A_{2,1} & A_{c,\bar{o}} & A_{2,3} & A_{2,4} \\ 0 & 0 & A_{\bar{c},o} & 0 \\ 0 & 0 & A_{4,3} & A_{\bar{c},\bar{o}} \end{bmatrix} \times \begin{bmatrix} x_c(t) \\ x_{c,\bar{o}}(t) \\ x_{\bar{c},o}(t) \\ x_{\bar{c},\bar{o}}(t) \end{bmatrix} + \begin{bmatrix} B_c \\ B_{c,\bar{o}} \\ 0 \\ 0 \end{bmatrix} (r(t) - y(t))$$

$$u(t) = \begin{bmatrix} C_c & 0 & C_{\bar{c},o} & 0 \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_{c,\bar{o}}(t) \\ x_{\bar{c},o}(t) \\ x_{\bar{c},\bar{o}}(t) \end{bmatrix} \quad (45)$$

so $[x_c^T(t) \;\; x_{c,\bar{o}}^T(t) \;\; x_{\bar{c},o}^T(t) \;\; x_{\bar{c},\bar{o}}^T(t)]^T \in \mathbb{R}^n$. According to this theorem, this nth order controller has the same transfer-matrix as (7), so we can apply the techniques of the previous section. However, it is convenient to mention three aspects of this development:

First, the selection of matrices in (45) (apart from those belonging to (7)) is constrained by the conditions given in Section 3.4, according to which the uncontrollable eigenvalues of the controller (i.e. the eigenvalues of $A_{\bar{c},o}$ and $A_{\bar{c},\bar{o}}$) must belong to S, and its unobservable eigenvalues (i.e. the eigenvalues of $A_{c,\bar{o}}$ and $A_{\bar{c},\bar{o}}$) must belong to $\bar{S}$. If the eigenvalues of $A_{\bar{c},\bar{o}}$ are distinct between themselves and with respect to the rest of the controller and the original closed-loop state matrix, H_o (given by the plant and the original controller), then they cannot belong simultaneously to S and $\bar{S}$, since

$$\sigma(H) = S \cup \bar{S} = \sigma(H_o) \cup \sigma(A_{c,\bar{o}}) \cup \sigma(A_{\bar{c},o}) \cup \sigma(A_{\bar{c},\bar{o}}) \quad (46)$$

but only $A_{\bar{c},\bar{o}}$ has these eigenvalues with multiplicity 1. This means that there is no internal equivalence for a controller with a matrix $A_{\bar{c},\bar{o}}$ whose eigenvalues are distinct between themselves and with respect to the rest of the controller and H_o. [This condition is equivalent to the proposition that if the eigenvalues of $A_{\bar{c},\bar{o}}$ are simple, there are no internal equivalences for the resulting controller.] Hence, it is convenient to restrict realisation (45) to

$$\begin{bmatrix} \dot{x}_c(t) \\ \dot{x}_{c,\bar{o}}(t) \\ \dot{x}_{\bar{c},o}(t) \end{bmatrix} = \begin{bmatrix} A_c & 0 & A_{1,3} \\ A_{2,1} & A_{c,\bar{o}} & A_{2,3} \\ 0 & 0 & A_{\bar{c},o} \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_{c,\bar{o}}(t) \\ x_{\bar{c},o}(t) \end{bmatrix} + \begin{bmatrix} B_c \\ B_{c,\bar{o}} \\ 0 \end{bmatrix} (r(t) - y(t))$$

$$u(t) = \begin{bmatrix} C_c & 0 & C_{\bar{c},o} \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_{c,\bar{o}}(t) \\ x_{\bar{c},o}(t) \end{bmatrix} \quad (47)$$

Second, it is important to note that even if it is mathematically feasible to consider unstable matrices $A_{c,\bar{o}}$, $A_{\bar{c},o}$ or $A_{\bar{c},\bar{o}}$, this is not advisable from a control point of view, since they give an internally unstable closed-loop (see, e.g. [2]), whose internal signals may be unbounded.

Third, even after satisfying conditions of Section 3.4, the selection of matrices of (47) is not unique. This can give rise to many external equivalences of the original transfer-matrix controller. This is shown in the next example.

Example 2. Controller of lower order than the plant: Consider a plant given by

$$G_0(s) = \begin{bmatrix} \dfrac{2}{(s+1)(s+2)} & \dfrac{-0.5}{s+2} \\ \dfrac{0.5}{s+1} & \dfrac{6}{(s+2)(s+3)} \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cc} -1 & 0 & 0 & 0 & 2.291 & 0 \\ 0 & -2 & 0 & 0 & 2.236 & 0.2608 \\ 0 & 0 & -3 & 0 & 0 & 5 \\ 0 & 0 & 0 & -2 & 0 & -4.59 \\ \hline 0.8729 & -0.8944 & 0 & 0.0581 & 0 & 0 \\ 0.2182 & 0 & -1.2 & -1.307 & 0 & 0 \end{array}\right] \quad (48)$$

and a controller given by

$$C(s) = \begin{bmatrix} \dfrac{0.192}{s} & 0 \\ 0 & \dfrac{0.352}{s} \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0.192 & 0 & 0 & 0 \\ 0 & 0.352 & 0 & 0 \end{array}\right] \quad (49)$$

which assigns the closed-loop poles at $\{-3.430;\ -2.154;\ -0.8793 \pm j0.2020;\ -0.3285 \pm j0.2092\}$.

To apply the previous technique it is necessary to extend the controller model (49) to one of order 4.

If an observable subsystem of order 2 is included in the controller, with eigenvalues at -4 and -5, this yields, for instance, to

$$\dot{\boldsymbol{x}}_c(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix} \boldsymbol{x}_c(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} (\boldsymbol{r}(t) - \boldsymbol{y}(t))$$

$$\boldsymbol{u}(t) = \begin{bmatrix} 0.192 & 0 & 0 & 0 \\ 0 & 0.352 & 0 & 0 \end{bmatrix} \boldsymbol{x}_c(t) \tag{50}$$

For this controller, the eigenvalues of the associated pseudo-Hamiltonian matrix correspond to the closed-loop poles plus the unobservable subsystem eigenvalues incorporated to the controller; that is

$$\sigma(\boldsymbol{H}) = \{-3.430, -2.154, -0.8793 \pm j0.2020, -0.3285 \pm j0.2092, -4, -5\} \tag{51}$$

According to the necessary existence conditions, the unobservable controller eigenvalues, -4 and -5, must belong to $\bar{S}$ (i.e. they cannot belong to S). Then, by choosing

$$S = \{-3.430, -2.154, -0.3285 \pm j0.2092\} \tag{52}$$

matrices $\boldsymbol{J}$ and $\boldsymbol{K}$ are given by

$$\boldsymbol{J} = \begin{bmatrix} -1.277 & 0.6050 \\ -3.044 & 0.6286 \\ -0.2971 & -3.161 \\ 0.2532 & 2.134 \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} -0.4294 & 0.1252 & 0.03990 & 0.1440 \\ 0.09440 & -0.02210 & 0.2137 & 0.4613 \end{bmatrix} \tag{53}$$

The combination of a state-feedback and a full-order observer, based on matrices $\boldsymbol{J}$ and $\boldsymbol{K}$ of (53), has a matrix transfer which coincides with that of the original controller (49).

If, instead of (52), we choose

$$S = \{-0.8793 \pm j0.2020, -0.3285 \pm j0.2092\} \tag{54}$$

then

$$\boldsymbol{J} = \begin{bmatrix} 3.881 & 0.5261 \\ 0.3252 & 0.1529 \\ -0.06572 & 0.7090 \\ 0.5790 & -3.206 \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} 0.1463 & -1.114 & 0.0502 & -0.01801 \\ 0.03329 & -0.02168 & -0.9806 & -0.3223 \end{bmatrix} \tag{55}$$

The last alternative consists of choosing

$$S = \{-3.430, -2.154, -0.8793 \pm j0.2020\} \tag{56}$$

to obtain

$$\boldsymbol{J} = \begin{bmatrix} -3.534 & -0.1652 \\ -5.058 & 0.2108 \\ 4.529 & -3.974 \\ -6.674 & 3.157 \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} -0.1510 & 0.09003 & -0.005384 & -0.02069 \\ -0.4200 & 0.05130 & 0.1680 & 0.2975 \end{bmatrix} \tag{57}$$

The transfer functions for combinations (55) and (57) coincide with that of the original controller (within a numerical precision of ten significant digits for the transfer function coefficients).

If the necessary existence conditions are ignored, for instance, by choosing

$$S = \{-4, -5, -3.430, -2.154\} \tag{58}$$

the Riccati equation has no solution. Hence, there are no matrices $\boldsymbol{J}$ and $\boldsymbol{K}$ associated to this choice of S.

Assume that a second-order stable unobservable and uncontrollable subsystem is incorporated to the original controller, with eigenvalues at -4 and -5. In this case, one possible solution can be built using the augmented controller

$$\dot{\boldsymbol{x}}_c(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix} \boldsymbol{x}_c(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} (\boldsymbol{r}(t) - \boldsymbol{y}(t))$$

$$\boldsymbol{u}(t) = \begin{bmatrix} 0.192 & 0 & 0 & 0 \\ 0 & 0.352 & 0 & 0 \end{bmatrix} \boldsymbol{x}_c(t) \tag{59}$$

The pseudo-Hamiltonian matrix in this case has the same spectrum than before. Nevertheless, the resulting Riccati equation has no solution for any of the three possible choices of S

$$\begin{aligned} S &= \{-3.430, -2.154, -0.3285 \pm j0.2092\} \\ S &= \{-0.8793 \pm j0.2020, -0.3285 \pm j0.2092\} \\ S &= \{-3.430, -2.154, -0.8793 \pm j0.2020\} \end{aligned} \tag{60}$$

Note. The previous treatment can also be applied to consider controllers of higher order than the plant order. To this end, we simply add uncontrollable or unobservable poles to the plant, to equate the controller order, and we then proceed as before.

5 Conclusions

The problem of the internal equivalence has been studied using the idea, due to Luenberger, that almost any system may be seen as an observer of another. This means that, in principle, it is possible to transform the state of the controller to be an estimation of the plant state. With this idea, we have found that the equivalence problem may be solved constructively in terms of the invertible solutions of an asymmetric Riccati equation, each one of which determines uniquely a state-feedback and a full-order observer whose combination is internally equivalent to the given controller. This is the main contribution of this paper.

We have analysed the solution of the asymmetric Riccati equation related to the problem, and we have noted that in many cases the solutions of the equation are defined by a combination of n out of the $2n$ eigenvalues of the pseudo-Hamiltonian matrix of the equation, which corresponds to the closed-loop state matrix. In other words, the number of degrees-of-freedom related to the equivalence problem in the MIMO case is similar to those of the SISO case. However, as we have mentioned in the paragraph following Example 1, there are MIMO cases where the number of degrees-of-freedom is actually less than what is expected for SISO cases at first sight. It is possible, in principle, that this number may be infinite, but to the knowledge of the authors there are no cases where this happens.

In order to know whether there exists at least one internal equivalence for a given classical loop, it helps to develop some conditions on the existence of invertible solutions of an asymmetric Riccati equation. One necessary condition for this has been presented in this section, but sufficient conditions have yet to be found, which is a consequence of the lack of results on asymmetric Riccati equation in the specialised literature.

Finally, the method has been extended to the case when the controller has lower order than the plant. To do this, uncontrollable and/or unobservable stable modes must be added to the controller model, so that its order is equal to that of the plant. Using the necessary condition for equivalence, it has been shown that, to obtain an equivalence, the controller cannot have modes which are, both uncontrollable and unobservable. A simple extension of this idea solves the equivalence problem for controllers of higher order than the plant order.

Future research could investigate necessary and sufficient conditions for the equivalence to exist. Another interesting topic relates to biproper controllers and their observed-state-feedback equivalences. Preliminary results show that this problem can be solved using singular perturbation theory [11].

6 Acknowledgment

The authors gratefully acknowledge the support received through grants USM-230425 and Fondecyt-1040313.

7 References

1 Yuz, J., and Salgado, M.: 'From classical to state-feedback-based controllers', *IEEE Control Syst. Mag.*, 2003, **23**, pp. 58–67

2 Goodwin, G.C., Graebe, S.F., and Salgado, M.E.: 'Control system design' (Prentice-Hall, 2001)

3 Kailath, T.: 'Linear systems' (Prentice-Hall, 1980)

4 Luenberger, D.G.: 'Observing the state of a linear system', *IEEE Trans. Military Electron.*, 1964, **8**, pp. 74–80

5 Luenberger, D.G.: 'Observers for multivariable systems', *IEEE Trans. Autom. Control*, 1966, **11**, pp. 190–197

6 Luenberger, D.G.: 'An introduction to observers', *IEEE Trans. Autom. Control*, 1971, **16**, pp. 596–602

7 Rojas, C.R.: 'Sobre la equivalencia entre el control mediante matrices de transferencia y el control por variables de estado'. Tesis de Magíster, Universidad Técnica Federico Santa María, 2004

8 Horn, R.A., and Johnson, C.R.: 'Matrix analysis' (Cambridge University Press, 1985)

9 Chen, C.T.: 'Linear system theory and design' (3rd edn, Oxford University Press, 1999)

10 Kučera, V.: 'Riccati equations and their solutions' 'The control handbook' (CRC Press 1995), pp. 595–606

11 Rojas, C.R.: 'On the equivalence between transfer matrix control and observed-state feedback control: the biproper case'. Technical Report, Universidad Técnica Federico Santa María, 2005

12 Medanic, J.: 'Geometric properties and invariant manifolds of the riccati equation', *IEEE Trans. Autom. Control*, 1982, **27**, pp. 670–677

8 Appendix A. Some theorems

Theorem 1 (Solutions of a generalised Riccati equation): Consider the generalised Riccati equation

$$\boldsymbol{TA} - \boldsymbol{DT} + \boldsymbol{TBT} - \boldsymbol{C} = 0 \qquad (61)$$

where $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D} \in \mathbb{R}^{n\times n}$, and define the pseudo-Hamiltonian matrix

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \qquad (62)$$

whose $2n$ eigenvalues are λ_i ($i = 1, \ldots, 2n$). By joining n generalised right eigenvectors of $\boldsymbol{H}$ as column vectors, create matrix $[\boldsymbol{F}^T\ \boldsymbol{G}^T]^T$, where $\boldsymbol{F} = [f_{1,1} \cdots f_{1d_1} \cdots f_{k,1} \cdots f_{k,d_k}]$ and $\boldsymbol{G} = [g_{1,1} \cdots g_{1,d_1} \cdots g_{k,1} \cdots g_{k,d_k}]$, such that:

1. $\boldsymbol{F}$ is invertible.
2. Eigenvalues associated with chosen generalised right eigenvectors come in complex conjugates, and there is an equal number of generalised right eigenvectors for each eigenvalue and its conjugate.
3. Chosen generalised right eigenvectors which belong to the same eigenvalue of $\boldsymbol{H}$ are a Jordan chain.

Then $\boldsymbol{T} = \boldsymbol{GF}^{-1}$ is a real solution of the Riccati equation (61), and for this solution it holds that $\boldsymbol{A} + \boldsymbol{BT}$ has the eigenvalues $\{\lambda_i\}$ associated with the chosen generalised right eigenvectors. Conversely, every real solution of (61) may be written as $\boldsymbol{T} = \boldsymbol{GF}^{-1}$, for some choice of generalised right eigenvectors of $\boldsymbol{H}$ satisfying the three previous properties.

Proof: This is a simple extension of Exercise 3.4-10 of [3]. □

Theorem 2 (Existence of invertible solutions of a Riccati equation): Consider matrix $\boldsymbol{T} = \boldsymbol{GF}^{-1}$ (where $\boldsymbol{F}$ and $\boldsymbol{G}$ are defined as in Theorem 1). Let S be the set of eigenvalues associated with the generalised right eigenvectors of matrix $\boldsymbol{H}$ (defined in (62)) which conform the columns of $[\boldsymbol{F}^T\ \boldsymbol{G}^T]^T$, and let $\bar{S}$ be the set of the remaining eigenvalues of $\boldsymbol{H}$ (even if some of them are also in S). A necessary condition for T to be an invertible solution of the Riccati equation (61) is that S must contain all uncontrollable eigenvalues of pairs $(\boldsymbol{A}, \boldsymbol{B})$ and $(\boldsymbol{D}, \boldsymbol{C})$, and that $\bar{S}$ must contain all unobservable eigenvalues of pairs $(\boldsymbol{A}, \boldsymbol{C})$ and $(\boldsymbol{D}, \boldsymbol{B})$.

Proof: This is a variant of Theorem 2 of [12]. Suppose that $\boldsymbol{T}$ is a solution of (61). Consider then the invertible transformation matrix

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{T} & \boldsymbol{I} \end{bmatrix} \qquad (63)$$

From the inverse of a block triangular matrix (see Exercise A.22 of [3]), we have that, after some algebra

$$\boldsymbol{U}^{-1}\boldsymbol{HU} = \begin{bmatrix} \boldsymbol{A} + \boldsymbol{BT} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{D} - \boldsymbol{TB} \end{bmatrix} \qquad (64)$$

But $\sigma(\boldsymbol{U}^{-1}\boldsymbol{HU}) = \sigma(\boldsymbol{H})$, so we deduce that $\sigma(\boldsymbol{H}) = \sigma(\boldsymbol{A} + \boldsymbol{BT}) \cup \sigma(\boldsymbol{D} - \boldsymbol{TB})$ (see Theorem 5 of [7]). Also, from Theorem 1 we know that $\sigma(\boldsymbol{A} + \boldsymbol{BT}) = S$, so $\sigma(\boldsymbol{D} - \boldsymbol{TB}) = \bar{S}$.

Suppose that λ is an uncontrollable eigenvalue of pair $(\boldsymbol{A}, \boldsymbol{B})$. This means, by the PBH test, that there exists a left eigenvector $\boldsymbol{v} \in \mathbb{C}^n$ of $\boldsymbol{A}$, associated with an eigenvalue

λ of $\boldsymbol{A}$, such that $\boldsymbol{v}^T\boldsymbol{B} = 0$. Therefore

$$\boldsymbol{v}^T(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{T}) = \boldsymbol{v}^T\boldsymbol{A} + \boldsymbol{v}^T\boldsymbol{B}\boldsymbol{T} = \lambda\boldsymbol{v}^T \tag{65}$$

so $\lambda \in \sigma(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{T}) = S$.

Suppose now that λ is an unobservable eigenvalue of pair $(\boldsymbol{D}, \boldsymbol{B})$. This means, by the PBH test, that there exists a right eigenvector $\boldsymbol{v} \in \mathbb{C}^n$ of $\boldsymbol{D}$, associated with an eigenvalue λ of $\boldsymbol{D}$, such that $\boldsymbol{B}\boldsymbol{v} = 0$. Therefore

$$(\boldsymbol{D} - \boldsymbol{T}\boldsymbol{B})\boldsymbol{v} = \boldsymbol{D}\boldsymbol{v} - \boldsymbol{T}\boldsymbol{B}\boldsymbol{v} = \lambda\boldsymbol{v} \tag{66}$$

so $\lambda \in \sigma(\boldsymbol{D} - \boldsymbol{T}\boldsymbol{B}) = \bar{S}$.

Consider now another invertible transformation matrix

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \tag{67}$$

which satisfies

$$\boldsymbol{V}^{-1}\boldsymbol{H}\boldsymbol{V} = \begin{bmatrix} \boldsymbol{D} & \boldsymbol{C} \\ \boldsymbol{B} & \boldsymbol{A} \end{bmatrix} \tag{68}$$

Therefore $\boldsymbol{V}^{-1}\,\boldsymbol{H}\boldsymbol{V}$ is the pseudo-Hamiltonian matrix of the Riccati equation

$$\tilde{\boldsymbol{T}}\boldsymbol{D} - \boldsymbol{A}\tilde{\boldsymbol{T}} + \tilde{\boldsymbol{T}}\boldsymbol{C}\tilde{\boldsymbol{T}} - \boldsymbol{D} = 0 \tag{69}$$

and if $\boldsymbol{T}$ is an invertible solution of (61), then (61) may be pre-multiplied and post-multiplied by $\boldsymbol{T}^{-1}$, yielding

$$\boldsymbol{A}\boldsymbol{T}^{-1} - \boldsymbol{T}^{-1}\boldsymbol{D} + \boldsymbol{B} - \boldsymbol{T}^{-1}\boldsymbol{C}\boldsymbol{T}^{-1} = 0 \tag{70}$$

so $\tilde{\boldsymbol{T}} = \boldsymbol{T}^{-1}$ is a solution of (69). This means that for every invertible solution of (61) there is an invertible solution of (69). Also, from Theorem 1 we have that if $\boldsymbol{T}$ is a solution of (61) formed from $S = \sigma(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{T})$, then $\tilde{\boldsymbol{T}} = \tilde{\boldsymbol{T}}^{-1}$ is a solution of (69) formed from set

$$\begin{aligned} \tilde{S} &= \sigma(\boldsymbol{D} + \boldsymbol{C}\tilde{\boldsymbol{T}}) = \sigma(\boldsymbol{T}[\boldsymbol{T}^{-1}\boldsymbol{D} + \boldsymbol{T}^{-1}\boldsymbol{C}\boldsymbol{T}^{-1}]) \\ &= \sigma(\boldsymbol{T}[\boldsymbol{A} + \boldsymbol{B}\boldsymbol{T}]\boldsymbol{T}^{-1}) = S \end{aligned} \tag{71}$$

By applying the same previous arguments on this matrix, we deduce that S contains all uncontrollable eigenvalues of pair $(\boldsymbol{D}, \boldsymbol{C})$, and that $\bar{S}$ contains all unobservable eigenvalues of pair $(\boldsymbol{A}, \boldsymbol{C})$. □

Theorem 3 (Controllability and observability relationships): Consider matrices $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}^{n\times m}$, $\boldsymbol{C} \in \mathbb{R}^{p\times n}$, $\boldsymbol{U} \in \mathbb{R}^{m\times q}$, and $\boldsymbol{V} \in \mathbb{R}^{r\times p}$, where $\boldsymbol{U}$ has linearly independent rows and $\boldsymbol{V}$ has linearly independent columns. Then the pair $(\boldsymbol{A}, \boldsymbol{B})$ is controllable if and only if so is the pair $(\boldsymbol{A}, \boldsymbol{B}\boldsymbol{U})$. Similarly, the pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable if and only if so is the pair $(\boldsymbol{A}, \boldsymbol{V}\boldsymbol{C})$.

Proof: See Theorem 16 of [7]. □