Design of Feedback Quantizers for Networked Control Systems

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Abstract

This paper deals with the design of feedback quantizers to encode plant output measurements in networked control systems with data-rate constrained channels. Starting form a nominal design made under the assumption of transparent communication links, we show how to design a feedback quantizer so as to systematically reduce the impact of quantization on closed loop performance. To obtain our results, we model quantization errors as additive white noise with a signal-to-noise ratio constraint. As a byproduct, we obtain a simple characterization of the minimal quantizer signal-to-noise ratio that allows one to design a feedback quantizers that guarantees stability. This bound depends only upon the plant and controller unstable poles. If the plant is strongly stabilizable, then the bound is consistent with the absolute minimal data-rate for stabilization obtained in previous work.

1 Introduction

Standard control theory deals with situations where the communication links between plant and controller can be regarded as transparent (see, e.g., [22,55]). There exist, however, cases where the links in a control system are far from being transparent and may become bottlenecks in the achievable performance. Control systems where this happens are collectively referred to as Networked Control Systems (NCS's) (see, e.g., [1,2,27,28] and the many references therein). Clearly, unless the channel characteristics are explicitly taken into account at the design stage, the performance of an NCS may be far from optimal and sometimes completely unsatisfactory. The main issues that need to be considered when dealing with NCS's include data-rate constraints (i.e., quantization), data loss and random delays. A unifying framework for the treatment of the general NCS design problem does not exist. Nevertheless, there has been significant progress in the study of specific situations that focus on subproblems. For example, data-rate constraints have been studied in [40,42,45,48,60] and design strategies to deal with quantization have been proposed in, e.g., [21,66]. The issue of data loss has been studied in [33,49,51], among many others, and delays have been considered in, e.g., [32,43,58,62].

In this paper we focus on linear time invariant (LTI) plant models, and concentrate on the effects of quantization on closed loop performance. Within this framework, a key result obtained in [40] relates the minimal data-rate which is necessary and sufficient to achieve stabilization of an unstable plant,

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to its poles in very simple way. This bound has been linked to information theoretic concepts where it has been given an interpretation akin to that of entropy (see, e.g., [37, 41, 48]). Furthermore, [7, 8] established that the minimal data-rate for stabilization is sometimes consistent with minimal signal-to-noise ratio requirements in standard one-degree-of-freedom control architectures that employ LTI controllers. More precisely, [7, 8] showed that, if the plant is defined in discrete time, has relative degree one and is minimum phase, then a Gaussian memoryless channel having a signal-to-noise ratio equal to the lower bound derived in [7, 8] would exhibit a channel capacity equal to the data-rate bound in [40].[1]

The results discussed above give absolute lower limits on the admissible channel data-rate which cannot be by-passed by any control law. It seems, however, quite difficult to obtain practical design guidelines from considerations such as those in [40, 41, 48]. This has motivated some researchers to move towards a simplified treatment of quantization. For example, [21] models quantization as a sector bound uncertainty and employs standard robust control tools. On the other hand, [66] uses a simple white noise model for quantization errors. The latter model for quantization has close connections to the signal processing literature, where it has been successfully used to design high performance quantization schemes (see, e.g., [4, 25, 30, 36, 50]).

In the present work we assume that a controller has already been designed under the assumption of transparent communication links. However, we subsequently extend the set-up by assuming that the control loop has to be implemented using a bit-rate limited channel in the plant to controller communication link. Thus, the plant output measurements have to be quantized prior to transmission. To that end, we borrow ideas from the signal processing literature and employ a feedback quantizer to encode the plant output (see, e.g., [30, 50]). Using a fixed signal-to-noise ratio additive noise model for quantization errors, we show how to design the feedback quantizer so as to systematically reduce the impact of quantization on closed loop performance, as measured by the tracking error variance. We show via simulations that our approach gives very good results even for bit rates as low as one bit per sample. We also study stability properties for this linear model. As a byproduct, we obtain a simple characterization of the minimal quantizer signal-to-noise ratio that allows one to design a feedback coding system that guarantees stability. This result is expressed in terms of the plant and controller unstable poles only. For stable controllers, and regardless of the plant zeros or relative degree, our results suggest a minimal data-rate for stabilization that is consistent[2] with the bound in [40].

The idea of designing coding schemes to embellish given controller designs is not new. For example, our previous work documented in [23] considers a coding scheme that turns out to be a special case of the one considered here. On the other hand, [53] considers the same coding architecture as the one studied in this paper, but the design procedure in [53] assumes that quantization effects are *relatively small.* The methodology used in the current paper does not require this assumption. Also, the stability analysis included in the current paper goes beyond the results of [23, 53]. Another related line of work has been developed in [9, 10, 34]. The latter work presents a precise deterministic stability analysis when the coding system is constrained to be a Δ-modulator (or variations thereof;[3] see, e.g., [30]), but does not address performance issues. Another recent publication closely related to the current paper is [38]. In that work, the authors propose a coding architecture similar to the one in this paper, but restrict the quantizers to have infinitely many levels and a prespecified quantization step. The latter assumptions are not needed here. Interestingly, the optimal coder in [38] (which focuses on minimizing

[1] If the plant has non-minimum phase zeros or a relative degree greater than one, then the analysis in [7, 8] suggests that a data-rate strictly greater than the aforementioned bound is necessary.

[2] in the same sense as the results in [8].

[3] i.e., unlike our proposal, the general multi-bit case is not treated in [9, 10, 34]

a time domain functional) turns out to have a structure that is a special case of the architecture considered here.

The remainder of this paper is organized as follows: Section 2 presents the notation employed in the paper. Section 3 describes the NCS architecture of interest and derives a linear model that is suitable for analysis and synthesis using linear system theoretical tools. Section 4 studies stability properties of the linear model, while Section 5 presents the proposed design procedure. Section 6 documents a simulation study. Concluding remarks are included in Section 7.

2 Notation

We use standard vector space notation for signals, i.e., x denotes $\{x(k)\}_{k\in\mathbb{N}_0}$. We also use z as both the argument of the z-transform and as the forward shift operator, where the meaning is clear from the context. Given any matrix X, $(X)^H$ and $(X)^T$ denote conjugate transposition and transposition, respectively. Given any complex scalar x, $|x|$ and $\bar{x}$ denotes magnitude and complex conjugation, respectively.

The set of all discrete time real rational transfer functions is denoted by $\mathcal{R}$. We define six subsets of $\mathcal{R}$ as follows: $\mathcal{R}_p$ contains all proper transfer functions, $\mathcal{R}_{sp}$ contains all strictly proper transfer functions, $\mathcal{RH}_\infty$ contains all stable and proper transfer functions, $\mathcal{U}_\infty$ contains all matrices in $\mathcal{RH}_\infty$ that have inverses in $\mathcal{RH}_\infty$, $\mathcal{RH}_2$ contains all stable and strictly proper transfer functions, $\mathcal{RH}_2^\perp$ contains all transfer functions that have only poles outside the unit circle and are either proper or improper. For any $A(z) \in \mathcal{R}$ we define $A(z)^\sim \triangleq A(z^{-1})^T$. We say that $A(z) \in \mathcal{R}$ is unitary if and only if $A(z)^\sim A(z) = I$. We also define $\{A(z)\}|_{z=\infty} \triangleq A(\infty) = \lim_{z\to\infty} A(z)$.

Every $A(z) \in \mathcal{R}$ with no poles on the unit circle belongs to $\mathcal{L}_2$ in which case we define the 2$-$norm of $A(z)$ via (see, e.g., [39])

$$||A(z)||_2^2 \triangleq \text{trace}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} A(e^{j\omega})^H A(e^{j\omega})d\omega\right\}.$$

For each such $A(z)$, we can always find $A_\perp(z) \in \mathcal{RH}_2^\perp$ and $A_2(z) \in \mathcal{RH}_2$ such that $A(z) = A_\perp(z) + A_2(z)$ and, accordingly, $||A(z)||_2^2 = ||A_\perp(z)||_2^2 + ||A_2(z)||_2^2$ (see, e.g., [39]).

Any biproper $n \times 1$ transfer matrix $A(z) \in \mathcal{RH}_\infty$ admits an inner-outer factorization of the form

$$A(z) = A_i(z)A_o(z),$$

where $A_i(z) \in \mathcal{RH}_\infty$ is unitary (i.e., $A_i(z)$ is inner) and $A_o(z) \in \mathcal{RH}_\infty$ is a biproper scalar transfer function, that has no zeros in $|z| > 1$ (i.e., $A_o(z)$ is scalar and outer). Moreover, if $A(z)$ has no zeros on the unit circle, then $A_o(z) \in \mathcal{U}_\infty$ (see, e.g., [20]).

Given any wide sense stationary (wss) process x, we denote its power spectral density by $S_x(e^{j\omega})$, its variance by σ_x^2 and its standard deviation by σ_x. We note that if, in addition, x has an always positive rational spectrum, then we can always find a spectral factor $\Omega_x(z) \in \mathcal{U}_\infty$ such that $S_x(e^{j\omega}) = \Omega_x(e^{j\omega})\Omega_x(e^{j\omega})^H$, $\forall\omega \in [-\pi, \pi]$. We also recall the well known fact that $\sigma_x^2 = ||\Omega_x(z)||_2^2$ (see, e.g., [56]).

3 Coding for Networked Control Systems

In this paper, we will consider the NCS architecture depicted in Figure 1. In that figure, $G(z)$ is the SISO plant model, $C(z)$ is a SISO controller, y is the plant output, r is the reference signal, d_o

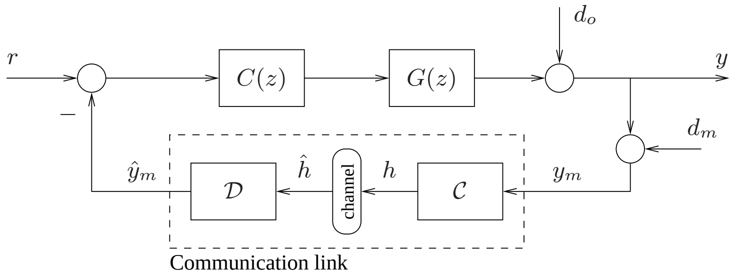

Figure 1: Considered networked control architecture.

models output disturbances and d_m corresponds to measurement noise. Unlike standard non-networked situations (see, e.g, [22, 55]), the feedback path in Figure 1 comprises a communication channel and a (source) coding/decoding system ($\mathcal{C}$ and $\mathcal{D}$).[4] The main focus of the current paper lies in designing this coding system, having performance in mind. To that end, we utilize as the performance assessment quantity the stationary variance of the tracking error e, defined via

$$e \triangleq r - y. \tag{1}$$

We next describe the assumptions that underly our subsequent analysis.

3.1 The nominal design

Since our aim is to design coding systems, we will assume that the controller $C(z)$ in Figure 1 has been already designed assuming transparent communication links.[5] The control loop formed by $C(z)$ and $G(z)$ when transparent communication links are in place (i.e., when $\hat{y}_m = y_m$ in Figure 1) will be referred to as the *nominal loop* (or nominal design).

For future reference we note that, in the nominal loop,

$$e = T_{we}(z)w, \tag{2}$$

where $w \triangleq \begin{bmatrix} r & d_o & d_m \end{bmatrix}^T$,

$$T_{we}(z) \triangleq \begin{bmatrix} S(z) & -S(z) & T(z) \end{bmatrix}, \tag{3}$$

and $S(z)$ and $T(z)$ are defined via

$$S(z) \triangleq \frac{1}{1 + G(z)C(z)}, \quad T(z) \triangleq 1 - S(z). \tag{4}$$

The transfer function $S(z)$ is the nominal loop sensitivity function and $T(z)$ is the nominal loop complementary sensitivity function (see [22]).

In the sequel, we will assume the following:

[4] In the sequel, we will use the term *coding system* (or just *coder*) to refer to both the encoder and decoder in Figure 1.

[5] This can, of course, be carried out using any standard design tool (see, e.g., [22, 55]).

Assumption 1 (Plant and nominal design) *The plant model belongs to $\mathcal{R}_{sp}$, whilst the controller, $C(z)$, belongs to $\mathcal{R}_p$, is non-zero and is such that the nominal loop is stable and well posed (in the standard sense; see, e.g., [22, 55]).*

The assumption that the nominal loop is stable and well defined is, of course, sensible in our context where the coding system is designed *a-posteriori*. We assume that $G(z)$ is strictly proper for simplicity. In principle, this can be removed at the expense of additional technical care. On the other hand, the assumption of $C(z)$ being non-zero discards non interesting situations, where the nominal loop is such that $G(z)$ is left in open loop (i.e., uncontrolled).

We end this section with a description of the exogenous signals r, d_o and d_m.

Assumption 2 (Signals) *The signals r, d_o and d_m are mutually independent scalar zero mean wss processes, each having a rational power spectral density that, if not identically zero, admits a spectral factor in $\mathcal{U}_\infty$.*

We note that Assumption 2 is standard (see, e.g, [56]).

3.2 The coding system

In this paper, we will focus on error free bit-rate limited channels. As a consequence, the input to the channel, i.e., h (see Figure 1), must be quantized prior to transmission. To that end, we will consider a standard feedback quantizer as depicted in Figure 2 (also known as a noise shaping quantizer; see, e.g., [30, 50]). In that figure, $A(z), B(z)$ and $F(z)$ are filters in $\mathcal{R}_p$ that need to be designed and $\mathcal{Q}$ denotes a uniform quantizer (see, e.g., [25, 30]), i.e.,[6]

$$\mathcal{Q}(v(k)) \triangleq \text{sat}_V\left(\Delta \left\lfloor \frac{v(k)}{\Delta} \right\rfloor + \frac{\Delta}{2}\right), \tag{5}$$

where V is the quantizer dynamic range, $\Delta \triangleq 2V(L-1)^{-1}$ and L is the number quantization levels.

We recall that a quantizer is said to be *overloaded* if and only if the absolute value of its input is greater than its dynamic range, i.e., $|v(k)| > V$ for some $k \in \mathbb{N}_0$. If the quantizer does not overload, then the quantization noise, defined via

$$q \triangleq h - v, \tag{6}$$

is such that $|q(k)| \leq \frac{\Delta}{2}$ for every k.

As already mentioned in Section 3.1, we are interested in designing coding systems for pre-specified nominal designs. In this setting, it is natural to employ coding systems that, in the absence of channel artifacts, have unit transfer function. That is, we will utilize coding systems that achieve *perfect reconstruction*. In our case, the channel is assumed to be error-free and hence $\hat{h} = h$. As a consequence, it is straightforward to see from Figure 2 that[7]

$$\hat{y}_m = B(z)A(z)y_m + B(z)(1-F(z))q, \tag{7}$$

[6] $\text{sat}_V(x) \triangleq x$ if $|x| \leq V$ and $\text{sat}_V(x) \triangleq \frac{x}{|x|}V$ if $|x| > V$; $\lfloor x \rfloor$ denotes the integer part of x.

[7] We note that (7) holds irrespective of the nature of the quantization noise q. Of course, (7) is of no utility, unless q is guaranteed to have some appropriate properties.

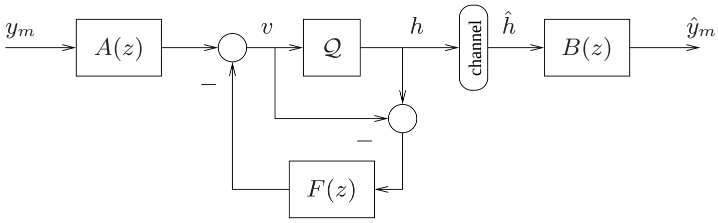

Figure 2: Considered coding and decoding system.

It follows from (7) that perfect reconstruction is tantamount to having $B(z) = A(z)^{-1}$ for every z. On the other hand, in order to have a properly defined feedback loop around the quantizer it is necessary to have a strictly proper $F(z)$ (see, e.g., Chapter 4 in [44]). We summarize the previous discussion as follows:

Constraint 1 (Structural constraints on the feedback quantizer) *The feedback quantizer filters are such that $B(z) = A(z)^{-1}$ and $F(z) \in \mathcal{R}_{sp}$.*

Quantization is a deterministic non-linear operation and hence, the exact analysis of quantized systems is difficult (see, e.g., [16, 26, 40, 47]). It has thus become standard, particularly in the signal processing literature (see, e.g., [5, 25, 30, 36, 50, 63]), to approximate quantization noise by an additive white noise source uncorrelated with the input of the quantizer. Here, we adopt this paradigm and assume the following:

Assumption 3 (Quantization noise model) *The quantization noise signal q (defined in (6)) is a sequence of i.i.d. random variables uniformly distributed in $\frac{1}{2}[-\Delta, \Delta)$, and uncorrelated with w.*

Note that we do not assume that the quantization noise is uncorrelated with v, which is certainly not the case since the quantization noise is fed-back to the input of the quantizer and, moreover, the coding system is inside the main feedback control loop. Instead, we adopt a milder assumption that requires only uncorrelatedness with the exogenous signals contained in w. We stress that the previous model is valid only if Δ is small enough, the quantizer does not overload and v has a smooth probability density (see, e.g., [4]). These conditions usually do not hold in the case of quantizers that are embedded in feedback loops (see, e.g., the discussion regarding stand alone feedback quantizers in [24]). Nevertheless, one can make use of dithered quantizers (see, e.g., [25, 65]) to render the model in Assumption 3 exact provided no overload occurs. Despite the above points, we will see in the simulation study included in Section 6, that, even if one employs a non-dithered uniform quantizer with as few as 2 levels, the predictions made using the simple model summarized in Assumption 3 are surprisingly accurate (see also simulation studies in [17, 23, 53]).

In order to guarantee that the quantizer does not overload, in principle one needs to consider infinite quantization levels (or assume that the quantizer input is deterministically bounded, which is seldom the case in a stochastic framework). In practice, it is standard to choose a dynamic range such that the probability of overload is negligible (see, e.g, [30]). Indeed, if v is wss and β is any positive real, then one can always find a finite α such that choosing $V = \alpha\sigma_v$ guarantees that the probability of overload is less than β; α is called the quantizer *loading factor*.[8] With such a choice for the overloading factor,

[8] For example, if $v(k)$ is Gaussian, then $\alpha = 4$ guarantees an overload probability of $6.33 \cdot 10^{-5}$.

it is immediate to see that

$$\frac{\sigma_v^2}{\sigma_q^2} = \frac{12 \cdot \sigma_v^2}{\Delta^2} = \frac{3}{\alpha^2}(L-1)^2, \tag{8}$$

where we have used the fact that, according to Assumption 3, $\sigma_q^2 = \frac{\Delta^2}{12}$. This justifies the following additional assumption:

Assumption 4 (Fixed signal-to-noise ratio) *For a fixed number of quantization levels, the variance of the quantization noise is proportional to the variance of the signal being quantized, i.e., the quantizer has a fixed signal-to-noise ratio γ defined via*

$$\gamma \triangleq \frac{\sigma_v^2}{\sigma_q^2}. \tag{9}$$

Assumption 4 is a key constraint. As mentioned before, it allows one to guarantee that the quantizer dynamic range is always properly scaled. In addition, it has a *regularizing* effect on the optimization based design of the coding system. Indeed, if this constraint were not in place (i.e., if q were assumed to have some prescribed statistics), then it would be optimal to choose $F(z) = 0$ and $A(z)^{-1} = \epsilon$ with $\epsilon \to 0$. This is, of course, not a sensible choice since $A(z)$ and σ_v^2 grow unbounded when $\epsilon \to 0$.

Remark 1 *We would like to stress that, in some situations, quantizer overload may become the dominant quantization effect in feedback schemes. Indeed, quantizer overload may trigger limit cycle oscillations that are, of course, not predicted by the linear model for quantization introduced above (see, e.g., [19,44,47]). As implied by Assumptions 3 and 4, we assume in this paper that quantizer overload is infrequent enough and, accordingly, that it has no significative effect on overall closed loop performance. (Careful design of the quantizer loading factor may act as a safeguard against quantizer overload.)*

Considering the model for quantization described above, together with the nominal loop description in Section 3.1, it is easy to derive the linear model shown in Figure 3 for the considered NCS. (Note that we have made the perfect reconstruction constraint explicit.) In Figure 3, q satisfies Assumptions 3 and 4, and r, d_o, d_m satisfy Assumption 2. We will refer to this model as *the linear model.* It will be the basis of the remainder of this paper.

4 Mean Square Stability

In this section we study stability properties of the linear model for the considered NCS derived in Section 3. In particular, we characterize all filters $F(z)$ and $A(z)$ that lead to stable linear models (in an appropriate sense) for a given quantizer signal-to-noise ratio γ. As a byproduct, we characterize the *minimal* quantizer signal to noise ratio γ that allows one to find $F(z)$ and $A(z)$ such that the resulting linear model is stable.

We begin by noting that, if x is a n-dimensional vector that contains the states of $C(z)$, $G(z)$, $A(z)$, $A(z)^{-1}$ and $F(z)$ (see Figure 3), then the evolution of x can be described by a linear state space model:

$$x(k+1) = Ax(k) + B_w w(k) + B_q q(k), \quad x(0) \triangleq x_o, \tag{10a}$$
$$v(k) = C_v x(k) + D_{wv} w(k) + D_{qv} q(k), \tag{10b}$$

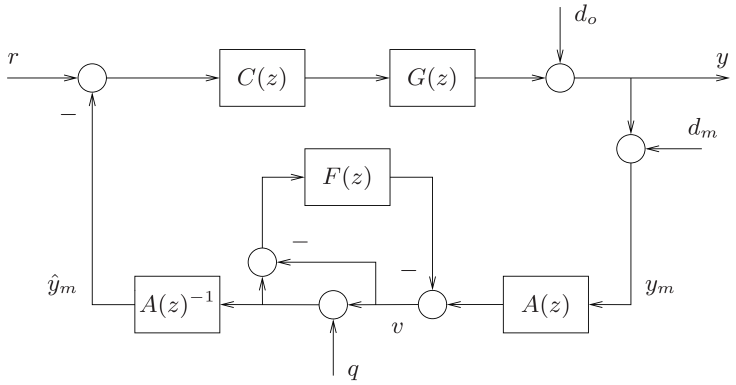

Figure 3: Linear model for considered networked situation.

where A, B_w, B_q, C_v, D_{wv} and D_{qv} are matrices of appropriate dimensions that depend on the particular realizations of $C(z), G(z), A(z), A(z)^{-1}$ and $F(z)$. Next, since we are considering a stochastic system, we need an appropriate notion of stability:

Definition 1 (Mean Square Stability [12, 13, 31]) *The linear system in* (10) *is Mean Square Stable (MSS*[9]*) if and only if there exist a finite $\mu_x \in \mathbb{R}^n$ and a finite $R_x \in \mathbb{R}^{n\times n}$, $R_x \geq 0$,*[10] *both not dependent on the initial state x_o, such that*[11]

$$\lim_{k\to\infty} \mu_x(k) = \mu_x, \quad \lim_{k\to\infty} \mathcal{E}\left\{ (x(k) - \mu_x(k))\,(x(k) - \mu_x(k))^H \right\} = R_x, \tag{11}$$

where $\mu_x(k) \triangleq \mathcal{E}\{x(k)\}$.

The next theorem gives necessary and sufficient conditions for MSS in our case:

Theorem 1 (Conditions for Mean Square Stability) *If Assumptions 1-4 hold, and x_o is an independent random variable with finite mean and finite variance matrix, then the linear model in Figure 3 is MSS if and only if $A(z) \in \mathcal{U}_\infty$, $F(z) \in \mathcal{RH}_2$ and*

$$\gamma > ||T(z) + S(z)F(z)||_2^2\,. \tag{12}$$

Proof: Define R_w as the variance matrix of w. Since the spectral factor of w, $\Omega_w(z)$, belongs to $\mathcal{RH}_\infty$, we lose no generality if we restrict attention to the case where $\Omega_w(z)\Omega_w(z)^\sim = R_w \geq 0$, for every z (i.e., if we assume that w is white noise). The general case employs the same arguments, but requires an augmented description of the system that has additional stable modes.

[9] By a slight abuse of notation we will also use MSS to refer to Mean Square Stability, where the meaning is clear from the context.

[10] $\geq$ stands for positive semi-definite.

[11] $\mathcal{E}\{\cdot\}$ stands for the expectation operator. We also note that the definition of limit implies that these quantities, if they exist, are unique.

Consider the state space description of the system under study given by (10). Standard results allow one to conclude (see, e.g., Chapter 4 in [56]) that under our working assumptions

$$\mu_x(k) = A^k \mathcal{E}\{x_o\} \tag{13}$$

$$R_x(k+1,k+1) = AR_x(k,k)A^H + \begin{bmatrix} B_w & B_q \end{bmatrix} R_{wq} \begin{bmatrix} B_w & B_q \end{bmatrix}^H, \tag{14}$$

where R_{wq} is the variance matrix of the vector $\begin{bmatrix} w(k) & q(k) \end{bmatrix}^T$ and

$$R_x(k,k) \triangleq \mathcal{E}\left\{ (x(k) - \mu_x(k)) (x(k) - \mu_x(k))^H \right\}. \tag{15}$$

Moreover, we also have that

$$\mu_v(k) = C_v \mu_x(k) \tag{16}$$

$$R_v(k,k) = C_v R_x(k,k) C_v^H + \begin{bmatrix} D_{wv} & D_{qv} \end{bmatrix} R_{wq} \begin{bmatrix} D_{wv} & D_{qv} \end{bmatrix}^H, \tag{17}$$

where $\mu_v(k)$ and $R_v(k,k)$ are defined as $\mu_x(k)$ and $R_x(k,k)$, but considering v instead of x.

- ($\Rightarrow$) If the NCS is MSS, then both μ_x and $R_x \geq 0$ are finite and unique. Therefore, (13) implies that A must be stable. On the other hand, we also see from (17) that $\lim_{k\to\infty} R_v(k,k)$, i.e., the stationary variance of v, say σ_v^2, must be positive semi-definite, finite and unique.

  Since the nominal loop is stable, a simple calculation shows that A being stable implies that both $A(z)$ and $A(z)^{-1}$ must be stable, and moreover, that $F(z)$ is stable. Of course, both $A(z)$ and $A(z)^{-1}$ must be proper and, on the other hand, $F(z)$ is constrained to be strictly proper (recall Constraint 1). Therefore, it follows that $A(z) \in \mathcal{U}_\infty$ and $F(z) \in \mathcal{RH}_2$.

  If A is stable, then it is easy to see that the stationary variance of v satisfies (see also Section 5.1)

$$\sigma_v^2 = ||A(z)T_{wy_m}(z)R_w||_2^2 + \sigma_q^2 \, ||T(z) + S(z)F(z)||_2^2, \tag{18}$$

  where $T_{wy_m}(z)$ is defined in (30), and where we have used the fact that w is uncorrelated with q and is such that $\Omega_w(z) = R_w$. Using the definition of γ in (18) yields

$$\sigma_v^2 = \frac{||A(z)T_{wy_m}(z)R_w||_2^2}{\gamma - ||T(z) + S(z)F(z)||_2^2}. \tag{19}$$

  Therefore, we conclude that, provided A is stable, σ_v^2 being positive semi-definite, finite and unique is equivalent to (12).

- ($\Leftarrow$) Since the nominal loop is stable, $F(z) \in \mathcal{RH}_2$ and $A(z) \in \mathcal{U}_\infty$, we have that A is stable. Therefore, it follows from (13) that μ_x is finite, unique and well defined.

  If (12) holds, then the facts deduced when proving the sufficiency part of this theorem imply that σ_v^2 is positive semi-definite, unique and finite. Using the definition of γ, it then follows that σ_q^2 is positive semi-definite, finite and unique. Therefore, R_{wq} is positive semi-definite, unique and finite and, since A is guaranteed to be stable (see above), then we have that the Lyapunov equation that describes the limiting value of $R_x(k,k)$ in (14) admits a finite, unique and positive semi-definite solution (see, e.g., Section 21.1 in [67]). Therefore, we have proven that R_x is as required. This completes the proof.

□□□

The condition for MSS given in Theorem 1 is deceivingly simple. This is due to the fact that the nominal loop is assumed stable and we are focusing on coding systems that achieve perfect reconstruction (recall Section 3.2). It is relevant to note that (12) does not depend on $A(z)$. Therefore, one can easily characterize the greatest lower bound on γ that allows one to guarantee MSS:

Theorem 2 (Minimal signal-to-noise ratio for MSS) *If Assumptions 1-4 hold, and x_o is an independent random variable with finite mean and finite variance matrix, then there exist filters $A(z)$ and $F(z)$ that allow one to guarantee MSS if and only if*

$$\gamma > \gamma_{\text{inf}} \triangleq \inf_{F(z)\in\mathcal{RH}_2} ||T(z)+S(z)F(z)||_2^2 = \left(\prod_{i=1}^{n} |p_i|^2\right) - 1, \tag{20}$$

where $\{p_i\}_{i\in\{1,\cdots,n\}}$ denotes the set of unstable poles of $G(z)C(z)$.

Proof: It suffices to compute $\inf_{F(z)\in\mathcal{RH}_2} ||T(z)+S(z)F(z)||_2^2$ (see Theorem 1). To that end, we employ the techniques described in detail in, e.g., [8, 39, 64]. We first note that $F(z)\in\mathcal{RH}_2 \Leftrightarrow Q(z)\triangleq zF(z)\in\mathcal{RH}_\infty$. Define

$$\xi_S(z) = \prod_{i=1}^{n_p^+} \frac{1-z\bar{p}_i}{z-p_i}, \tag{21}$$

where $\{p_i\}_{i\in\{1,\cdots,n_p^+\}}$ denotes the set of non-minimum phase zeros of $S(z)$ that lie strictly outside the unit circle (i.e., the unstable poles of $G(z)C(z)$ outside the unit circle). It is clear that $\xi_S(z)\in\mathcal{RH}_2^\perp$, is unitary, and is such that the transfer function $\xi_S(z)S(z)$ belongs to $\mathcal{RH}_\infty$, is biproper and has as non-minimum phase zeros the zeros on the unit circle of $S(z)$ (i.e., the poles on the unit circle of $G(z)C(z)$). It thus follows that

$$\begin{aligned}
||T(z)+S(z)F(z)||_2^2 &= \left|\left|1-S(z)+S(z)z^{-1}Q(z)\right|\right|_2^2 \\
&= \left|\left|\xi_S(z)-\xi_S(z)S(z)+\xi_S(z)S(z)z^{-1}Q(z)\right|\right|_2^2 \\
&= ||\xi_S(z)-\xi_S(0)||_2^2 + ||z\xi_S(0)-z\xi_S(z)S(z)+\xi_S(z)S(z)Q(z)||_2^2 \\
&= ||\xi_S(z)-\xi_S(0)||_2^2 + ||\xi_S(0)-\xi_S(\infty)||_2^2 + \\
&\qquad ||z\left(\xi_S(z)S(z)-\xi_S(\infty)\right)-\xi_S(z)S(z)Q(z)||_2^2,
\end{aligned} \tag{22}$$

where we have used orthogonal decompositions in $\mathcal{L}_2$, the fact that both $\xi_S(z)$ and z are unitary, the fact that Assumption 1 implies $S(\infty)=1$, and basic properties of the 2-norm. By construction, $z\left(\xi_S(z)S(z)-\xi_S(\infty)\right)\in\mathcal{RH}_\infty$ and $\xi_S(z)S(z)$ is invertible in $\mathcal{RH}_\infty$ except for zeros on the unit circle. Elementary results (see, e.g., Chapter 6 in [64]) allow one to conclude from (22) that

$$\begin{aligned}
\inf_{F(z)\in\mathcal{RH}_2} ||T(z)+S(z)F(z)||_2^2 &= ||\xi_S(z)-\xi_S(0)||_2^2 + ||\xi_S(0)-\xi_S(\infty)||_2^2 \\
&= 1+2\xi_S(0)^2-2\xi_S(0)\xi_S(\infty)+\xi_S(\infty)^2 \\
&\qquad -2\,\mathrm{Re}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\xi_S(e^{j\omega})\xi_S(0)^H d\omega\right\}.
\end{aligned} \tag{23}$$

Use of the Residue Theorem and some simple algebra yields the desired result. □□□

Theorem 2 states a precise condition that the quantizer signal-to-noise ratio γ has to satisfy in order to be able to find a coding system that, when inserted in the feedback path of a stable nominal loop, guarantees the MSS of the resulting linear model. The bound on γ depends only on the unstable poles of $G(z)C(z)$, i.e., on the unstable poles of the plant and controller. If the plant model is strongly stabilizable (i.e., can be stabilized using a stable controller; see, e.g., [18]), then employing a stable controller in the nominal loop allows one to find a feedback coder capable of stabilizing the resulting linear model if and only if

$$\gamma > \left(\prod_{i=1}^{n_G} |p_{G_i}|^2 \right) - 1, \tag{24}$$

where $\{p_{G_i}\}_{i\in\{1,\cdots,n_G\}}$ denotes the set of unstable poles of $G(z)$. We note that the same conclusion applies if the controller is stable except for poles on the unit circle (e.g., controllers with integral action).

If we fix $F(z) = 0$, then γ must satisfy $\gamma > ||T(z)||_2^2$, which is a fixed constraint in our framework. If it were possible to redesign the controller under the constraint $F(z) = 0$, then one can use the results in [8] to establish that the admissible signal-to-noise ratio must satisfy

$$\gamma > \left(\prod_{i=1}^{n_G} |p_{G_i}|^2 \right) - 1 + \Delta_G, \tag{25}$$

where Δ_G is non-negative and depends on the non-minimum phase zeros and on the relative degree of the plant model $G(z)$ ($\Delta_G = 0$ if and only if $G(z)$ is minimum phase and has relative degree equal to one). We thus conclude that the inclusion of the proposed coding system allows one to reduce the requirements on the quantizer signal-to-noise ratio, at least for strong stabilizable plants (regardless of the plant zeros or relative degree). This reduction may be very significative if, e.g., the plant has high relative degree. This is an important indication of the benefits that coding brings to networked control situations. A question that remains open, however, is whether or not there exist different coding architectures that allow one to recover (24) for any plant.

Remark 2 (Relationship to prior work) *In [8] it is proved that* (25) *is the minimal signal-to-noise ratio that allows one to find one-degree-of-freedom controllers that stabilize a given LTI plant model over an additive noise channel with a* power constraint. *In a second step, the authors show that a Gaussian memoryless channel, with a signal-to-noise ratio γ that satisfies* (25)*, would have a capacity $\mathcal{C}_{ap}$ (see, e.g., [14]) that satisfies*

$$\mathcal{C}_{ap} = \frac{1}{2}\log_2(1+\gamma) > \sum_{i=1}^{n_G} \log_2|p_{G_i}| + \frac{1}{2}\log_2\left(1 + \frac{\Delta_G}{\prod_{i=1}^{n_G}|p_{G_i}|^2}\right) \geq \sum_{i=1}^{n_G}\log_2|p_{G_i}| \triangleq R_{\text{inf}}, \tag{26}$$

where R_{inf} is the minimal data-rate which is necessary and sufficient to stabilize an LTI system over an error-free bit-rate limited channel [40]. Equality in (26) *is achieved if and only if the plant is minimum phase and has a relative degree equal to one. These results* suggest *that signal-to-noise ratio requirements in LTI one degree-of-freedom control loops are, for a restricted class of plants, consistent with the minimal data-rate requirements of [40]. If the plant has non-minimum phase zeros, or has* *a relative degree larger than one, then* $\Delta_G > 0$ *(see* (25)*). It thus follows that, in these situations, data-rate requirements suggested by signal-to-noise ratio considerations may be more demanding than those in [40].*

Our results can be applied to the channel model in [8] as well, provided error free feedback (with a unit delay) is available from the channel output to the channel input.[12] *When doing so, it turns out that* (24) *is consistent (in the sense described above) with the minimal data-rate derived in [40]. Our results holds even if the plant model has arbitrary relative degree and arbitrary zeros, as long as* $G(z)$ *is strongly stabilizable. We thus conclude that, within the LTI framework, the use of feedback coding is key to achieve* (24) *and, accordingly, key to make signal-to-noise ratio requirements consistent with the results in [40]. We stress that the issue of existence of feedback from the channel output to the channel input is inconsequential to the set-up used in [40] because the channel is error free, as in our case. (Note that the assumption of channel feedback has been explicitly made for NCS's with stochastic channels (see, e.g., [3, 59, 61]) again recovering the results in [40]. If channel feedback is removed from the analysis of [59, 61] then the minimal data-rates for stabilization obtained do not necessarily coincide with those in [40] (see Section VI in [60])).*

5 Design for Performance

In this section we go beyond stability and focus on how to actually design a feedback coding system that minimizes the impact that the communication channel has on closed loop performance, as measured by the steady state variance of the tracking error.

5.1 Problem definition

The purpose of this section is to define the performance goals of interest in a precise way. To that end, we consider the linear model in Figure 3. Straightforward analysis reveals that the tracking error obeys

$$e = T_{we}(z)w + T(z)A(z)^{-1}(1 - F(z))q. \tag{27}$$

Therefore, if the linear model is MSS and Assumptions 2 and 3 hold, then the stationary variance of e exists and is given by

$$\sigma_e^2 = ||T_{we}(z)\Omega_w(z)||_2^2 + \sigma_q^2 \left|\left|T(z)A(z)^{-1}(1 - F(z))\right|\right|_2^2, \tag{28}$$

where $\Omega_w(z)$ is a spectral factor of the power spectral density of w. Since Assumption 4 holds, σ_q^2 is not a given constant; indeed, it depends on the variance of v. Proceeding as above (and using the same assumptions), it follows from Figure 3 that

$$v = A(z)T_{wy_m}(z)w - (T(z) + S(z)F(z))q, \tag{29}$$

where

$$T_{wy_m}(z) \triangleq \begin{bmatrix} T(z) & S(z) & -S(z) \end{bmatrix}, \tag{30}$$

[12] The authors of [8] exclude channel feedback from their setting.

and, accordingly,

$$\sigma_v^2 = ||A(z)T_{wy_m}(z)\Omega_w(z)||_2^2 + \sigma_q^2\, ||T(z) + S(z)F(z)||_2^2\,. \tag{31}$$

Using (9) and (31) in (28), it follows that

$$\sigma_q^2 = \frac{||A(z)T_{wy_m}(z)\Omega_w(z)||_2^2}{\gamma - ||T(z) + S(z)F(z)||_2^2}, \tag{32}$$

$$\sigma_e^2 = ||T_{we}(z)\Omega_w(z)||_2^2 + \frac{||A(z)T_{wy_m}(z)\Omega_w(z)||_2^2 \left|\left|T(z)A(z)^{-1}(1-F(z))\right|\right|_2^2}{\gamma - ||T(z) + S(z)F(z)||_2^2}. \tag{33}$$

We note that MSS of the linear model guarantees that $\gamma - ||T(z)+S(z)F(z)||_2^2 > 0$ and, consequently, $\sigma_q^2 \geq 0$, as expected.

We note that, since $C(z)$ is assumed to be given, the choice of the coding parameters (i.e., $A(z)$ and $F(z)$) affects only the second term in (33), which we denote as

$$J(A(z), F(z)) \triangleq \frac{||A(z)T_{wy_m}(z)\Omega_w(z)||_2^2 \left|\left|T(z)A(z)^{-1}(1-F(z))\right|\right|_2^2}{\gamma - ||T(z) + S(z)F(z)||_2^2}. \tag{34}$$

Accordingly, we can state the problem of interest as follows:

Problem 1 (Main problem) *Given a fixed $\gamma \in (\gamma_{\inf}, \infty)$, a controller $C(z)$ and a plant $G(z)$ that satisfy Assumption 1, and exogenous signals satisfying Assumption 2, find J_{opt} defined via*

$$J_{opt} \triangleq \inf_{\substack{A(z)\in\mathcal{U}_\infty \\ F(z)\in\mathcal{RH}_2 \\ ||T(z)+S(z)F(z)||_2^2<\gamma}} J(A(z), F(z)) \tag{35}$$

and filters $A(z)$ and $F(z)$ that achieve J_{opt} (or approximate J_{opt} arbitrarily well).

We note that all constraints in the formulation of Problem 1 stem from MSS considerations, as discussed in Theorems 1 and 2. We will use the term *admissible* $A(z)$ (resp. admissible $F(z)$) to refer to a filter $A(z)$ (resp. $F(z)$) that satisfies the constraints in Problem 1.

Problem 1 is non-trivial. Indeed, the much simpler problem of designing $A(z)$ and $F(z)$ so as to minimize the steady state variance of $\hat{y}_m - y_m$, when $G(z) = C(z) = 0$ and $d_m = r = 0$ has been only recently solved exactly (see [17]). This is quite surprising given the fact that feedback quantizers have been studied extensively (see, e.g., [30, 50, 57]). Unfortunately, the technique employed in [17] does not seem to yield an explicit characterization of the solution in the present situation. Instead of pursuing that line of reasoning here, we will derive an iterative approach that is guaranteed to yield performance that is arbitrarily close to optimum.

Before describing the proposed design procedure, we note that the following holds:

Fact 1 (Asymptotic behavior of J_{opt}) *Assume that the conditions of Problem 1 hold. Then:*

1. *If $\gamma \to \gamma_{\inf}$, then $J_{opt} \to \infty$ (unless all exogenous signals have zero spectral density, in which case $J(A(z)F(z)) = 0$ for every admissible $A(z)$ and $F(z)$).*

2. If $\gamma \to \infty$, then $J_{opt} \to 0$.

Proof:

1. By definition of J_{opt}, we have that $\gamma \to \gamma_{\text{inf}} \Rightarrow ||T(z) + S(z)F(z)||_2^2 \to \gamma_{\text{inf}}$. Thus $J_{opt} \to \infty$ unless either $||T(z)A(z)(1 - F(z))||_2^2 = 0$ or $||A(z)T_{wy_m}(z)\Omega_w(z)||_2^2 = 0$. Since $A(z) \in \mathcal{U}_\infty$ and Assumption 1 holds, the result follows upon noting that $\inf_{F(z)\in\mathcal{RH}_2} ||T(z)A(z)(1 - F(z))||_2^2 > 0$.

2. Fix $A(z) \in \mathcal{U}_\infty$ and $F(z) \in \mathcal{RH}_2$. In these conditions, $\gamma \to \infty \Rightarrow J \to 0$ and hence, $J_{opt} \to 0$.

□□□

As a consequence of Fact 1, we will omit from our subsequent presentation an explicit analysis of the cases $\gamma \to \infty$ or $\gamma \to \gamma_{\text{inf}}$. (The reader can easily verify that the results below are consistent with Fact 1 by letting $\gamma \to \infty$ or $\gamma \to \gamma_{\text{inf}}$.)

5.2 Choosing $A(z)$

We begin by showing how to choose $A(z)$, when an admissible $F(z)$ is given. To that end we define, for any given admissible $F(z)$,[13]

$$J_{opt}^1(F(z)) \triangleq \inf_{A(z)\in\mathcal{U}_\infty} J(A(z), F(z)), \tag{36}$$

$$A_{opt}^{F(z)}(z) \triangleq \arg \inf_{A(z)\in\mathcal{U}_\infty} J(A(z), F(z)). \tag{37}$$

The next theorem characterizes both $J_{opt}^1(F(z))$ and $A_{opt}^{F(z)}(z)$.

Theorem 3 (Optimal $A(z)$ for a given $F(z)$) *Assume that the conditions of Problem 1 hold and consider a fixed admissible $F(z)$. If $\Omega_w(z)$ is not identically zero, then:*

1. The minimal value of J is given by

$$J_{opt}^1(F(z)) = \frac{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|T(e^{j\omega})(1 - F(e^{j\omega}))\right| \sqrt{T_{wy_m}(e^{j\omega})\Omega_w(e^{j\omega})\Omega_w(e^{j\omega})^H T_{wy_m}(e^{j\omega})^H}\, d\omega\right)^2}{\gamma - ||T(z) + S(z)F(z)||_2^2}. \tag{38}$$

2. The corresponding optimal $A(z)$ satisfies

$$\left|A_{opt}^{F(z)}(e^{j\omega})\right|^4 = \alpha \frac{\left|T(e^{j\omega})(1 - F(e^{j\omega}))\right|^2}{T_{wy_m}(e^{j\omega})\Omega_w(e^{j\omega})\Omega_w(e^{j\omega})^H T_{wy_m}(e^{j\omega})^H}, \quad \forall\omega \in [-\pi, \pi], \tag{39}$$

where α is any arbitrary positive real.

[13] See comments in Appendix on the notation arg inf.

Proof: The definition of the 2−norm allows one to conclude that, for every $X(z) \in \mathcal{R} \cap \mathcal{L}_2$, the following identities hold:

$$||X(z)||_2^2 = \left|\left|X(z^{-1})^T\right|\right|_2^2 = \left|\left|\sqrt{X(z^{-1})^T X(z)}\right|\right|_2^2. \quad (40)$$

Both (38) and (39) follow using (40) and the Cauchy Schwartz inequality in (34) (note that (39) is always well defined if Assumption 1 holds, and $\Omega_w(z)$ is not identically zero). To complete the proof we note that (39) is a condition on the magnitude of the infimal filter $A(z)$. Thus, $A_{opt}^{F(z)}$ can always be approximated, to any desired degree of accuracy, by a rational filter in $\mathcal{U}_\infty$ as required. □□□

Remark 3 *Of course, assuming that $\Omega_w(z)$ is not identically zero does not hinder the generality of Theorem 3 (see Part 1 in Fact 1).*

The characterization of $A_{opt}^{F(z)}(z)$ given by Theorem 3, although explicit, is usually not satisfied by any transfer function in $\mathcal{U}_\infty$. This is due to the fact that, except in very special cases, the $4th$ root of the right hand side in (39) is irrational. Nevertheless, as mentioned in the proof of Theorem 3, it is always possible to find a filter in $\mathcal{U}_\infty$ that achieves a performance that is as close as desired to $J_{opt}^1(F(z))$.[14] In practice, it is usually enough to consider reasonably low order filters to approximate $A_{opt}^{F(z)}(z)$ (see also [23]).

5.3 Choosing $F(z)$

In this section we address the problem of choosing $F(z)$ when an admissible $A(z)$ is given. Consistent with the notation introduced before,

$$J_{opt}^2(A(z)) \triangleq \inf_{\substack{F(z)\in\mathcal{RH}_2 \\ ||T(z)+S(z)F(z)||_2^2<\gamma}} J(A(z), F(z)) \quad (41)$$

denotes the minimal value of J when $A(z) \in \mathcal{U}_\infty$ is fixed. We also define

$$F_{opt}^{A(z)}(z) \triangleq \arg \inf_{\substack{F(z)\in\mathcal{RH}_2 \\ ||T(z)+S(z)F(z)||_2^2<\gamma}} J(A(z), F(z)). \quad (42)$$

We begin by noting that $J_{opt}^2(A(z))$ can be written in a simpler form as follows:

Fact 2 (Equivalent formulation for $J_{opt}^2(A(z))$) *Assume that the conditions of Problem 1 hold and consider a fixed $A(z) \in \mathcal{U}_\infty$. Then,*

$$J_{opt}^2(A(z)) = ||A(z)T_{wy_m}(z)\Omega_w(z)||_2^2 \inf_{\gamma_{\inf}\leq M<\gamma} \frac{1}{\gamma - M} \inf_{\substack{F(z)\in\mathcal{RH}_2 \\ J_2(F(z))=M}} J_1(F(z)), \quad (43)$$

where

$$J_1(F(z)) \triangleq \left|\left|T(z)A(z)^{-1}(1-F(z))\right|\right|_2^2, \quad J_2(F(z)) \triangleq ||T(z)+S(z)F(z)||_2^2. \quad (44)$$

[14] This is, of course, consistent with the definition of $A_{opt}^{F(z)}$ and the definition of arg inf (see Appendix).

Proof: Using the definition of J and the fact that $A(z)$ is fixed, it is immediate to see that

$$J^2_{opt}(A(z)) = ||A(z)T_{wy_m}(z)\Omega_w(z)||_2^2 \inf_{\substack{F(z)\in\mathcal{RH}_2 \\ J_2(F(z))<\gamma}} \frac{J_1(F(z))}{\gamma - J_2(F(z))}. \tag{45}$$

Define a new real variable, M, constrained to belong to $[\gamma_{\inf}, \gamma)$. With this definition, elementary optimization results (see, e.g., Section 4.1.3 in [6]) allow one to write (45) as

$$J^2_{opt}(A(z)) = ||A(z)T_{wy_m}(z)\Omega_w(z)||_2^2 \inf_{\substack{F(z)\in\mathcal{RH}_2 \\ J_2(F(z))=M \\ \gamma_{\inf}\leq M<\gamma}} \frac{J_1(F(z))}{\gamma - M}, \tag{46}$$

where we have used the fact that, by definition of $\gamma_{\inf}$, $J_2(F(z)) \geq \gamma_{\inf}$ for any $F(z) \in \mathcal{RH}_2$. The result is now immediate. □□□

Fact 2 is key to derive the main result in this section. Namely, a one parameter characterization for $J^2_{opt}(A(z))$ and the corresponding optimal $F(z)$. Towards that goal, we begin by considering an auxiliary problem. Define the functional

$$L_\epsilon \triangleq \epsilon J_1(F(z)) + (1-\epsilon)J_2(F(z)), \tag{47}$$

where $\epsilon \in [0, 1]$, and

$$F_\epsilon(z) \triangleq \arg \inf_{F(z)\in\mathcal{RH}_2} L_\epsilon. \tag{48}$$

We have the following characterization of $F_\epsilon(z)$:

Lemma 1 (Solution to auxiliary problem) *Consider L_ϵ defined in* (47) *and suppose that Assumption 1 holds.*

1. *If $\epsilon \in (0, 1)$, then the infimum in* (48) *is achievable in $\mathcal{RH}_2$ and*

$$F_\epsilon(z) = 1 - P_{\epsilon,o}(\infty)P_{\epsilon,o}(z)^{-1}, \tag{49}$$

where $P_{\epsilon,o}(z) \in \mathcal{U}_\infty$ is an outer factor of

$$P_\epsilon(z) \triangleq \begin{bmatrix} \sqrt{\epsilon}\,\xi_T(z)T(z)A(z)^{-1} \\ \sqrt{1-\epsilon}\,\xi_S(z)S(z) \end{bmatrix} \tag{50}$$

and

$$\xi_T(z) \triangleq z^m \prod_{i=1}^{n_c^+} \frac{1 - z\bar{c}_i}{z - c_i}, \quad \xi_S(z) \triangleq \prod_{i=1}^{n_p^+} \frac{1 - z\bar{p}_i}{z - p_i}, \tag{51}$$

where m is the relative degree of $G(z)C(z)$, $\{c_i\}_{i\in\{1,\cdots,n_c^+\}}$ (resp. $\{p_i\}_{i\in\{1,\cdots,n_p^+\}}$) is the set of non-minimum phase zeros (resp. unstable poles) of $G(z)C(z)$ that lie strictly outside the unit circle.

2. If $\epsilon = 0$, then

$$F_0(z) = 1 - (\xi_S(z)S(z))^{-1}\,\xi_S(\infty) \tag{52}$$

and, if $\epsilon = 1$, then

$$F_1(z) = 1 - A(z)\,(\xi_T(z)T(z))^{-1}\,\left\{\xi_T(z)T(z)A(z)^{-1}\right\}\Big|_{z=\infty}. \tag{53}$$

3. If $\epsilon = 0$ (resp. $\epsilon = 1$), then the infimum in (48) is achievable in $\mathcal{RH}_2$, if and only if $G(z)C(z)$ has no poles (resp. zeros) on the unit circle.

Proof:

1. We will proceed as in the proof of Theorem 2 (see also [11]). As before, we define $Q(z) \in \mathcal{RH}_\infty$ via $F(z) \triangleq z^{-1}Q(z)$. It is easy to see from (22) that

$$\begin{aligned} L_\epsilon = (1-\epsilon)\left(\gamma_{\inf} + \|z\,(\xi_S(z)S(z) - \xi_S(\infty)) - \xi_S(z)S(z)Q(z)\|_2^2\right) + \\ \epsilon\left\|T(z)A(z)^{-1} - T(z)A(z)^{-1}z^{-1}Q(z)\right\|_2^2 \end{aligned} \tag{54}$$

Moreover, using the same procedure as in the aforementioned proof, it is also clear that

$$\begin{aligned} \left\|T(z)A(z)^{-1} - T(z)A(z)^{-1}z^{-1}Q(z)\right\|_2^2 &= \left\|z\xi_T(z)T(z)A(z)^{-1} - z\xi_T(z)T(z)A(z)^{-1}Q(z)\right\|_2^2 \\ &= \left\{\xi_T(z)T(z)A(z)^{-1}\right\}\Big|_{z=\infty}^2 + \\ \left\|z\left(\xi_T(z)T(z)A(z)^{-1} - \left\{\xi_T(z)T(z)A(z)^{-1}\right\}\Big|_{z=\infty}\right) - \xi_T(z)T(z)A(z)^{-1}Q(z)\right\|_2^2, \end{aligned} \tag{55}$$

where we have used the fact that the relative degree and non-minimum phase zeros of $T(z)$ are the relative degree and non-minimum phase zeros of $G(z)C(z)$, that $\xi_T(z)$ and z are unitary, and that, since $A(z) \in \mathcal{U}_\infty$, $\xi_T(z)$ is such that $\xi_T(z)T(z)A(z)^{-1}$ belongs to $\mathcal{RH}_\infty$, is biproper and has as non minimum phase zeros the zeros on the unit circle of $T(z)$ (i.e., the zeros on the unit circle of $G(z)C(z)$). From (54) and (55) it follows that

$$L_\epsilon = (1-\epsilon)\gamma_{\inf} + \epsilon\left\{\xi_T(z)T(z)A(z)^{-1}\right\}\Big|_{z=\infty}^2 + \hat{L}_\epsilon, \tag{56}$$

where

$$\hat{L}_\epsilon \triangleq \|W(z) - P_\epsilon(z)Q(z)\|_2^2, \tag{57}$$

$P_\epsilon(z)$ (defined in (50)) belongs to $\mathcal{RH}_\infty$, is biproper, and

$$W(z) \triangleq \begin{bmatrix} \sqrt{\epsilon}\,z\left(\xi_T(z)T(z)A(z)^{-1} - \left\{\xi_T(z)T(z)A(z)^{-1}\right\}\Big|_{z=\infty}\right) \\ \sqrt{1-\epsilon}\,z\,(\xi_S(z)S(z) - \xi_S(\infty)) \end{bmatrix} \in \mathcal{RH}_\infty. \tag{58}$$

Define the unitary matrix

$$\phi(z) \triangleq \begin{bmatrix} P_{\epsilon,i}(z)^\sim \\ I - P_{\epsilon,i}(z)P_{\epsilon,i}(z)^\sim \end{bmatrix}, \tag{59}$$

where $P_{\epsilon,i}(z)$ is an inner factor of $P_\epsilon(z)$ and $P_{\epsilon,o}(z)$ is the corresponding outer factor. We note that, since Assumption 1 holds, $P_\epsilon(z)$ has no zeros on the unit circle for $\epsilon \in (0,1)$. Thus, for those values of ϵ, $P_{\epsilon,o}(z) \in \mathcal{U}_\infty$.

Pre-multiplying the argument of $||\cdot||_2^2$ in (57) by $\phi(z)$ it is immediate to see that

$$\hat{L}_\epsilon = ||(I - P_{\epsilon,i}(z)P_{\epsilon,i}(z)^\sim)\, W(z)||_2^2 + ||P_{\epsilon,i}(z)^\sim W(z) - P_{\epsilon,o}(z)Q(z)||_2^2 . \tag{60}$$

A straightforward calculation shows that

$$P_{\epsilon,i}(z)^\sim W(z) = zP_{\epsilon,o}(z) - zP_{\epsilon,i}(z)^\sim P_\epsilon(\infty). \tag{61}$$

Therefore, orthogonal decompositions as those employed before allow one to write

$$\begin{aligned}\hat{L}_\epsilon = ||(I - P_{\epsilon,i}(z)P_{\epsilon,i}(z)^\sim)\, W(z)||_2^2 + ||P_{\epsilon,o}(\infty) - P_{\epsilon,i}(z)^\sim P_\epsilon(\infty)||_2^2 + \\ ||z\,(P_{\epsilon,o}(z) - P_{\epsilon,o}(\infty)) - P_{\epsilon,o}(z)Q(z)||_2^2 .\end{aligned} \tag{62}$$

Since $P_{\epsilon,o}(z) \in \mathcal{U}_\infty$ the result follows.

2. If $\epsilon \in \{0,1\}$, then (54) and (55) yield immediately the results.

3. The result follows upon noting that, by definition of $\xi_T(z)$ and $\xi_S(z)$, $(\xi_S(z)S(z))^{-1}$ (resp. $(\xi_T(z)T(z)A(z)^{-1})^{-1}$) belongs to $\mathcal{RH}_\infty$ if and only if $G(z)C(z)$ has no poles on the unit circle (resp. zeros on the unit circle).

□□□

The characterization of $F_\epsilon(z)$ given in Lemma 1 plays an essential role in our subsequent discussion. It is worth mentioning that the only critical step when calculating $F_\epsilon(z)$ is the inner-outer factorization of $P_\epsilon(z)$. Since $P_\epsilon(z)$ has no zeros at infinity (i.e., $P_\epsilon(z)$ is biproper), this factorization can be made with the aid of standard algorithms (see, e.g., [20, 46]).

The next theorem provides a characterization of the optimal $F(z)$ in terms of $F_\epsilon(z)$.

Theorem 4 (Optimal $F(z)$ for a fixed $A(z)$) *Assume that the conditions of Problem 1 hold and consider a fixed $A(z) \in \mathcal{U}_\infty$. Then,*

$$F_{opt}^{A(z)}(z) = F_{\epsilon^*}(z), \tag{63}$$

$$J_{opt}^2(A(z)) = ||A(z)T_{wy_m}(z)\Omega_w(z)||_2^2\, \frac{J_1(F_{\epsilon^*}(z)}{\gamma - J_2(F_{\epsilon^*}(z))}, \tag{64}$$

where[15]

$$\epsilon^* \triangleq \arg \min_{\epsilon\in(0,\hat{\epsilon})} \frac{J_1(F_\epsilon(z))}{\gamma - J_2(F_\epsilon(z))}. \tag{65}$$

In (65), $\hat{\epsilon}$ is defined as follows: If there does not exist $\epsilon \in (0,1]$ such that $J_2(F_{\epsilon_\gamma}(z)) = \gamma$, then $\hat{\epsilon} = 1$. Otherwise, $\hat{\epsilon} = \epsilon_\gamma$, where ϵ_γ is the unique real in $(0,1]$ such that $J_2(F_{\epsilon_\gamma}(z)) = \gamma$.

[15] We define ϵ^* using arg min instead of arg inf to stress that the infimum in (65) is actually achieved in $(0,\hat{\epsilon})$.

Proof: We will use the alternative formulation for J in Fact 2.

1. We first show how to solve an auxiliary problem related to the inner optimization problem in (43). Consider the problem:

$$\inf_{\substack{F(z)\in\mathcal{RH}_2\\ J_2(F(z))\leq M}} J_1(F(z)). \tag{66}$$

The well-known KKT conditions for this problem (see, e.g., [6,35]) allow one to conclude that the optimal $F(z)$, say $F_{aux}(z)$, (if it exists) is a critical point of $\lambda_1 J_1(F(z)) + \lambda_2 J_2(F(z))$, where $\lambda_1 + \lambda_2 > 0$, $\lambda_1, \lambda_2 \geq 0$ and, moreover, $\lambda_2(J_2(F_{aux}(z)) - M) = 0$. It is immediate to see that this is equivalent to saying that $F_{aux}(z)$ is a critical point of L_ϵ (see (47)), with $\epsilon \in [0,1]$ and $(1-\epsilon)(J_2(F_{aux}(z)) - M) = 0$.

L_ϵ is a strictly convex functional (and so are J_1 and J_2). Hence, it has an unique critical point given by $F_\epsilon(z)$ (see (48)). Moreover, the set of points in the J_1 versus J_2 plane defined by $F_\epsilon(z)$, when ϵ ranges from zero to one, is the set of Pareto optimal points of the multi-objective problem of minimizing simultaneously J_1 and J_2 (see, e.g., [6,15]). This set is a (strictly) convex and decreasing function when J_1 is seen as function of J_2. Clearly, the Pareto optimal point corresponding to $\epsilon = 1$ (resp. $\epsilon = 0$) is such that J_1 is minimum (resp. J_2 is minimum). Thus, by definition of Pareto optimal point, the minimum of J_1, when $J_2 \leq M$ is achieved when $J_2 = M$, provided $J_2(F_1(z)) \geq M$. If $J_2(F_1(z)) < M$, then the minimum J_1 is achieved when $J_2 = J_2(F_1(z))$. As a consequence, the optimal solution of the auxiliary problem is given by $F_{aux}(z) = F_{\epsilon_M}(z)$, where, provided $J_2(F_1(z)) \geq M$, ϵ_M belongs to $[0,1]$ and is such that $J_2(F_{\epsilon_M}(z)) = M$ (note that convexity ensures that, in this case, ϵ_M is unique). On the other hand, if $J_2(F_1(z)) < M$, then $\epsilon_M = 1$.

2. We next show how to exploit the above reasoning to prove the result. We first note that, since $F_1(z)$ optimizes J_1, it is of no use to consider values of M such that $J_2(F_1(z)) < M$ (note also that $M \to \gamma_{\inf}$ is optimal for the first optimization problem in (43) and, accordingly, constraining $J_2(F_1(z))$ to be greater than or equal to M does not impede the minimization of J). Thus, $F_{opt}(z)$ satisfies

$$F_{opt}(z) = \arg \inf_{\substack{\gamma_{\inf}\leq M<\gamma\\ M\leq J_2(F_1(z))}} \frac{1}{\gamma - M} \inf_{\substack{F(z)\in\mathcal{RH}_2\\ J_2(F(z))=M}} J_1(F(z)), \tag{67}$$

Using Part 1 it follows that $F_{opt}(z) = F_{\epsilon_{M^*}}(z)$ with M^* such that

$$M^* = \arg \inf_{\substack{\gamma_{\inf}\leq M<\gamma\\ M\leq J_2(F_1(z))}} \frac{J_1(F_{\epsilon_M}(z))}{\gamma - J_2(F_{\epsilon_M}(z))}, \tag{68}$$

where ϵ_M is guaranteed to exist in $[0,1]$ (and to be unique). A key feature of this problem is that the Pareto optimal points of the auxiliary problem considered in Part 1 do not depend on M. Thus, varying M in $[\gamma_{\inf}, \min\{\gamma, J_2(F_1(z))\}]$ is equivalent to just varying ϵ_M in $[0, \min\{1, \epsilon_\gamma\}]$ (if ϵ_γ [defined in the body of this Theorem] does not exist, then pick $\epsilon_\gamma = 1$). It should be clear that the structure of the problem is such that $M^* \in (\gamma_{\inf}, \min\{\gamma, J_2(F_1(z))\})$. Thus, it suffices to consider $\epsilon_M \in (0, \min\{1, \epsilon_\gamma\})$. As a consequence, the result follows.

□□□

Theorem 4 provides a one parameter characterization of the optimal $F(z)$ and the corresponding minimal cost $J^2_{opt}(A(z))$, for any admissible $A(z)$. The scalar parameter ϵ^* can be found using any standard line search procedure and, as such, its calculation embodies no additional difficulties. This is reinforced by the fact that the search for ϵ^* is made over $(0,1)$ (and that ϵ^* actually exists in $(0,1)$) which is precisely the range of values of ϵ for which $F_\epsilon(z)$ is always defined in $\mathcal{RH}_2$ (see Lemma 1).

5.4 Design procedure and final remarks

In this section we show how to use the results in Sections 5.2 and 5.3 to design a feedback coding system in an iterative fashion. Of course, one can always choose to fix one of the coder filters (trivial choices are $A(z) = 1$ or $F(z) = 0$) and then use Theorem 3 or 4 to design the free parameter. Obviously, this choice will limit the achievable performance. To exploit the full potential of a feedback coding system we suggest that one uses an iterative algorithm such as the following:

Algorithm 1 (Iterative design procedure) *For a given plant and controller satisfying Assumption 1, and a given $\gamma > \gamma_{\inf}$, proceed as follows:*

- ***Initialization:***
  - *Pick a tolerance $\rho > 0$, a transfer function $A_0(z) \in \mathcal{U}_\infty$ and a transfer function $F_0(z) \in \mathcal{RH}_2$ that is admissible.*
  - *Set $A(z) = A_0(z)$, $F(z) = F_0(z)$. Set $V(0) = J(A(z), F(z))$, fix $A(z)$ (or, alternatively, fix $F(z)$) and set $k = 0$.*
- ***Repeat** ($(k+1)^{th}$ **iteration**):*
  - *Set $k = k + 1$.*
  - *If at the $(k-1)^{th}$ iteration $A(z)$ was fixed, then use Theorem 4 to obtain $F^{A(z)}_{opt}(z)$. Set $F(z) = F^{A(z)}_{opt}(z)$ and $V(k) = J(A(z), F(z))$. Fix $F(z)$.*
  - *If at the $(k-1)^{th}$ iteration $F(z)$ was fixed, then use Theorem 3 to obtain $A^{F(z)}_{opt}(z)$. Set $A(z) = A^{F(z)}_{opt}(z)$ and $V(k) = J(A(z), F(z))$. Fix $A(z)$.*
- ***Until:** $|V(k) - V(k-1)|\, V(k-1)^{-1} < \rho$.*

It should be clear that it is not certain that Algorithm 1 will converge to the global minimum of J. Nevertheless, it is easy to see that, by definition of $A^{F(z)}_{opt}$ and $F^{A(z)}_{opt}$, the algorithm reduces the value of J at each iteration. Therefore, Algorithm 1 converges, necessarily, to a local minimum. Thus, we suggest to use multiple starting points so as to find the global minimum. A procedure for getting a good starting point is mentioned below.

In general, $A^{F(z)}_{opt}(z) \neq 1$ and $F^{A(z)}_{opt}(z) \neq 0$. Thus, fixing $A(z)$ or $F(z)$ and optimally choosing the other filter, will obviously provide a coding system that enhances closed loop performance when compared with a non-coded networked situation.[16] It is also clear that the use of Algorithm 1 allows

[16] Provided both situations us the same channel, the same quantizer and the same loading factor.

one to design coding systems that will always outperform coding systems that have been designed using the guidelines in our earlier work described in [23]. This is a consequence of the fact that [23] constrains $F(z)$ to be identically zero.

A more interesting discussion arises if one compares the results in this paper with the results in [53]. In the latter work, it is assumed that γ is *sufficiently high* so as to be able to approximate J in (34) by

$$J_\infty \triangleq \frac{||A(z)T_{wy_m}(z)\Omega_w(z)||_2^2 \, \left|\left|T(z)A(z)^{-1}(1-F(z))\right|\right|_2^2}{\gamma}. \tag{69}$$

In order to minimize J_∞ it suffices to choose $A(z)$ so as to minimize $||A(z)T_{wy_m}(z)\Omega_w(z)||_2^2$ and, in a second stage, to choose $F(z)$ as the minimizer of $\left|\left|T(z)A(z)^{-1}(1-F(z))\right|\right|_2^2$ [53]. A problem with the above approach is that deciding, *a priori*, which γ's are high enough seems to be impossible. In particular, since the procedure in [53] does not take the constraint $\gamma - ||T(z)+S(z)F(z)||_2^2 > 0$ explicitly into account, the proposed choice for $F(z)$ may be not admissible or may be such that $J(A(z), F(z)) \to \infty$. (Needless to say, this drawback is explicitly avoided in the current paper.) It is also clear that choosing $A(z)$ as in [53] and, then, using Theorem 4 to choose $F(z)$ will always lead to a feedback coder that achieves a tracking error variance that is lower than the one achieved by the filters proposed in [53]. Of course, if the filters suggested in [53] are feasible, then they may provide a good starting point for Algorithm 1.

6 Design Example

This section documents a design study that illustrates the results in this paper. We consider a very simple case that, nevertheless, will allow us to present the main features of our proposal.

We consider a nominal loop with plant and controller given by

$$G(z) = \frac{1}{z-0.8}, \quad C(z) = \frac{z-0.8}{z-1}. \tag{70}$$

The measurement noise and output disturbance are assumed zero, whilst the reference is considered to have a power spectral density with spectral factor

$$\Omega_r(z) = \frac{0.02z}{z-0.9}. \tag{71}$$

The quantizer loading factor is fixed at 4 in all cases,[17] and the number of quantization levels, $L \triangleq 2^b$, ranges between $L = 2^1$ and $L = 2^8$. Since we do not consider the use of channel coding schemes (e.g., entropy coding; see [9, 14]), b corresponds to the rate at which data is sent through the channel (in [bit/sample]).

Figure 4 shows the steady state tracking error variance σ_e^2 (see (33)) as a function of the number of iterations in Algorithm 1 for two representative values of the quantizer signal-to-noise ratio: $\gamma = 1.6875$ and $\gamma = 9.1875$, which correspond to $b = 2$ and $b = 3$, respectively. Cases 1 and 2 refer to iterations that start with $A_0(z) = 1$ and $F_0(z) = 0$. In Case 1 we initially fixed $A(z)$, whereas in Case 2 we start fixing $F(z)$. Case 3 refers to iterations that start with the choices suggested in [53]. We note that γ has to be greater than 4.42 in order for the proposal in [53] to be admissible. (Accordingly, we omitted

[17] This corresponds to the well known 4σ rule; see [30].

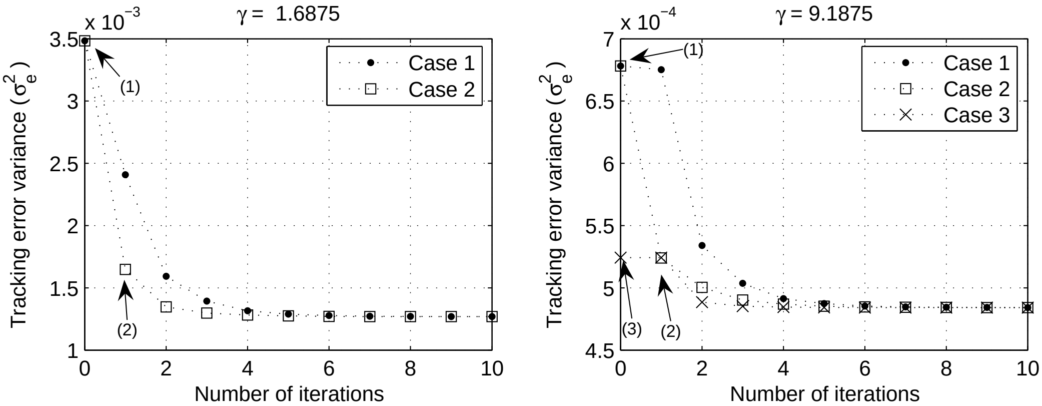

Figure 4: Tracking error as function of the number of iterations in Algorithm 1 (see text for details).

Case 3 in Figure 4 when $\gamma = 1.6875$.) It can be seen that rapid convergence of Algorithm 1 occurs and, more interestingly, that the limiting performance does not depend on the order in which the filters are calculated or on the initial condition. Thus, local minima related issues do not seem to play a role in this example.

In Figure 4 we have identified three points. The first of these (point (1)) refers to the performance achieved without coding ($F(z) = 0$ and $A(z) = 1$). The second (point (2)) refers to the performance achieved when employing the optimal coding system proposed in [23]. The third (point (3)) refers to the performance achieved using the approximately optimal filters described in [53].

The results show that coding is, indeed, necessary to achieve the best possible loop performance. (Compare point (1) with, e.g., the value of σ_e^2 for 10 iterations.) It is also possible to see that use of Algorithm 1 yields coding systems that perform better than our previous proposals in [23,53], which is consistent with the discussion at the end of Section 5.4. (Compare points (2) and (3) with the limiting value for σ_e^2.) It is also interesting to mention that, for $b > 4$, the performance provided by the filters in [53] is substantially closer to the limiting value of σ_e^2 than the case shown in Figure 4. This suggest, as mentioned before, that the filters in [53], when feasible, provide good starting points for the iterative procedure proposed here.

We end this section by studying the behavior of the tracking error variance as a function of the channel bit rate b. The results are presented in Figure 5, where "Nominal performance" refers to the performance achieved by the nominal loop (without quantization), "No coding (empirical)" refers to simulated results[18] when no coding is employed (i.e., when $A(z) = 1$ and $F(z) = 0$), "Opt. coding (empirical)" refers to simulated results obtained with the filters suggested by Algorithm 1 (after 10 iterations), and "Opt. coding (analytical)" refers to the corresponding predictions made using the simplified noise model for quantization. One can see that, as expected, the effects of quantization vanish as $b \to \infty$. Interestingly, the predictions made using our model turn out to be very accurate for every bit rate: indeed, for $b \geq 3$ the relative errors are of less than 1% and, for $b \in \{1, 2\}$, the relative errors are around 8%. (We note that $F(z) = 0$ turns out to be non-admissible for $b = 1$. Accordingly,

[18] All simulations use an actual undithered uniform quantizer with $L = 2^b$ levels. For each b, the results correspond to the average of 200 simulations (each one 10^5 samples long and using a different reference realization).

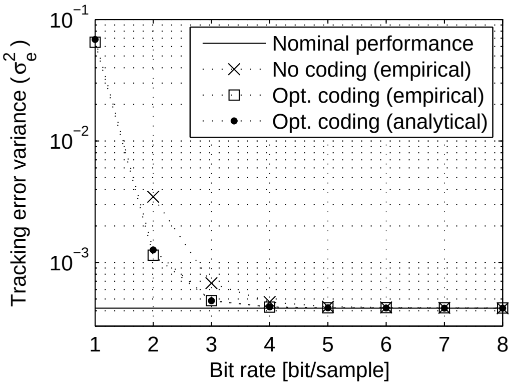

Figure 5: Tracking error as function of the channel bit rate.

we have omitted the non coded results for $b = 1$.)

7 Conclusions

This paper has presented a methodology to design feedback quantizers that encode plant output measurements in a networked control situation employing data-rate limited channels. Using a fixed signal-to-noise ratio additive noise model for quantization, we have shown how to iteratively design the parameters of a feedback coding system so as to minimize the impact of quantization on the closed loop tracking error. Our results show that feedback quantization schemes are beneficial when compared to simpler schemes documented in the literature. An interesting by-product of our results lies in the characterization of the smallest quantizer signal-to-noise ratio compatible with stabilization. We have shown that, for a given quantizer signal-to-noise ratio, the class of plants that are stabilizable when feedback coding is employed is significatively larger than the class of plants that are stabilizable when no coding is used. This result opens the door to investigating other LTI control and feedback coding architectures and the associated signal-to-noise ratio requirements.

A very interesting extension of the present work lies in addressing multiple-input multiple-output problems. In that case, it is worth exploring how networked architectures may help overcoming the well-known performance limitations that arise when constraining the structure of the controller (see, e.g., [29, 52]). A second immediate extension lies in the problem of joint controller and coder design. The study of how to apply similar ideas to the case of channels prone to data loss is also interesting (see preliminary work in [54]).

A Appendix

Consider a set $\mathcal{X}$ and a function J defined on $\mathcal{X} \subset \hat{\mathcal{X}}$ (and extensible to $\hat{\mathcal{X}}$). If $\inf_{X \in \mathcal{X}} J(X)$ exists and $\inf_{X \in \mathcal{X}} J(X)$ is achievable in $\mathcal{X}$, i.e., if $\exists\ X^* \in \mathcal{X}$ such that $J(X^*) = \inf_{X \in \mathcal{X}} J(X)$, then $X_{opt} \triangleq \arg\inf_{X \in \mathcal{X}} J(X) = X^*$. On the contrary, if $\nexists\ X^* \in \mathcal{X}$ such that $J(X^*) = \inf_{X \in \mathcal{X}} J(X)$, then X_{opt} defined as above should be understood as $X_{opt} = \lim_{n \to \infty} X_n$, where $\{X_n\}_{n \in \mathbb{N}}$ is a sequence in $\mathcal{X}$

(whose limit belongs to $\hat{\mathcal{X}}$) such that $\lim_{n\to\infty} J(X_n) \to \inf_{X\in\mathcal{X}} J(X)$. Therefore, if we write $X_{opt} = \hat{X}$ and $\hat{X} \notin \mathcal{X}$, it is implicit that one can find a sequence $\{X_n\}_{n\in\mathbb{N}}$ as above. In these cases, it is clear that one can always pick an $X \in \mathcal{X}$ such that $J(X)$ is as close to $\inf_{X\in\mathcal{X}} J(X)$ as desired.

References

[1] P. Antsaklis and J. Baillieul. Special issue on networked control systems. *IEEE Transactions on Automatic Control*, 49(9):1421–1423, September 2004.

[2] P. Antsaklis and J. Baillieul. Special issue on technology of networked control systems. *Proceedings of the IEEE*, 95(1):5–8, January 2007.

[3] L. Bao, M. Skoglund, and K.H. Johansson. On optimal system design for feedback control over noisy channels. In *IEEE International Symposium on Information Theory*, 2007. Nice, France.

[4] W.R. Bennet. Spectra of quantized signals. *Bell Syst. Tech. J.*, 27(4):446–472, 1948.

[5] H. Bölcskei and F. Hlawatsch. Noise reduction in oversampled filter banks using predictive quantization. *IEEE Transactions on Information Theory*, 47(1):155–172, January 2001.

[6] S. Boyd and L. Vandenberghe. *Convex Optimization.* Cambridge University Press, 2004.

[7] J.H. Braslavsky, R.H. Middleton, and J.S. Freudenberg. Feedback stabilisation over signal-to-noise ratio constrained channels. In *Proceedings of the American Control Conference*, Boston, USA, 2004.

[8] J.H. Braslavsky, R.H. Middleton, and J.S. Freudenberg. Feedback stabilization over signal-to-noise ratio constrained channels. *IEEE Transactions on Automatic Control*, 52(8):1391–1403, 2007.

[9] C. Canudas de Wit, J. Jaglin, and K.C. Vega. Entropy coding for networked controlled systems. In *Proceedings of the European Control Conference*, Kos, Greece, 2007.

[10] C. Canudas de Wit, F. Rodríguez, J. Fornés, and F. Gómez-Estern. Differential coding in networked controlled linear systems. In *Proceedings of the American Control Conference*, Minneapolis, USA, 2006.

[11] J. Chen, S. Hara, and G. Chen. Best tracking and regulation performance under control effort constraint. *IEEE Transactions on Automatic Control*, 48(8):1320–1380, August 2003.

[12] O.L.V. Costa and M.D. Fragoso. Stability results for discrete-time linear systems with Markovian jumping parameters. *Journal of Mathematical Analysis and Applications*, 179:154–178, 1993.

[13] O.L.V. Costa, M.D. Fragoso, and R.P. Marques. *Discrete Time Markov Jump Linear Systems.* Springer, 2005.

[14] T.M. Cover and J.A. Thomas. *Elements of Information Theory.* John Wiley and Sons, Inc., 2nd edition, 2006.

[15] N.O. Da Cunha and E. Polak. Constrained minimization under vector valued criteria in linear topological spaces. In A.V. Balakrishnan and L.W. Neustad, editors, *Mathematical Theory of Control, Proc. USC Conf.*, pages 96–108. Academic, New York, 1967.

[16] D.F. Delchamps. Stabilizing a linear system with quantized state feedback. *IEEE Transactions on Automatic Control*, 35(8):916–924, August 1990.

[17] M.S. Derpich, E.I. Silva, D.E. Quevedo, and G.C. Goodwin. On optimal perfect reconstruction feedback quantizers. *IEEE Transactions on Signal Processing*, 56(8):3871–3890, August 2008.

[18] J.C. Doyle, B.A. Francis, and A. Tannenbaum. *Feedback Control Theory.* Macmillan Publishing Company, New York, 1992.

[19] O. Feely. Nonlinear dynamics of discrete-time circuits: A survey. *Int. J. Circuit Theory Appl.*, 35(5-6):515–531, Sept.-Dec. 2007.

[20] B. Francis. *A Course on H_∞ Control Theory.* Springer, New York, 1987.

[21] M. Fu and L. Xie. The sector bound approach to quantized feedback control. *IEEE Transactions on Automatic Control*, 50(11):1698–1711, November 2005.

[22] G.C. Goodwin, S.F. Graebe, and M.E. Salgado. *Control System Design.* Prentice Hall, New Jersey, 2001.

[23] G.C. Goodwin, D.E. Quevedo, and E.I. Silva. Architectures and coder design for networked control systems. *Automatica*, 44(1):248–257, 2008.

[24] R.M. Gray. Quantization noise spectra. *IEEE Transactions on Information Theory*, 36(6):1220–1244, November 1990.

[25] R.M. Gray and T.G. Stockham (Jr.). Dithered quantizers. *IEEE Transactions on Information Theory*, 39(3):805–812, May 1993.

[26] C.S. Güntürk. One-bit sigma-delta quantization with exponential accuracy. *Commun. Pure Appl. Math.*, 56(11):1608–1630, 2003.

[27] J.P. Hespanha, P. Naghshtabrizi, and Y. Xu. A survey of recent results in networked control systems. *Proceedings of the IEEE*, 95(1):138–162, January 2007.

[28] D. Hristu-Varsakelis and W. Levine (Eds.). *Handbook of Networked and Embedded Systems.* Birkhäuser, 2005.

[29] H. Ishii and B.A. Francis. Stabilization with control networks. *Automatica*, 38:1745 – 1751, 2002.

[30] N. Jayant and P. Noll. *Digital Coding of Waveforms. Principles and Approaches to Speech and Video.* Prentice Hall, 1984.

[31] Y. Ji, H.J. Chizeck, X. Feng, and K.A. Loparo. Stability and control of discrete-time jump linear systems. *Control Theory Adv. Technol.*, 7(2):247–270, 1991.

[32] F. Lian, J. Moyne, and D. Tilbury. Modelling and optimal controller design of networked control systems with mutliple delays. *International Journal of Control*, 76(6):591–606, 2003.

[33] Q. Ling and M.D. Lemmon. Power spectral analysis of networked control systems with data dropouts. *IEEE Transactions on Automatic Control*, 49(6):955–960, June 2004.

[34] I. López, C.T. Abdallah, and C. Canudas de Wit. Gain-scheduling multi-bit delta-modulator for networked controlled system. In *Proceedings of the European Control Conference*, Kos, Greece, 2007.

[35] D. Luenberger. *Optimization by Vector Space Methods.* John Wiley and Sons, Inc., London, 1969.

[36] D. Marco and D.L. Neuhoff. The validity of the additive noise model for uniform scalar quantizers. *IEEE Transactions on Information Theory*, 51(5):1739–1755, May 2005.

[37] A. Matveev and A. Savkin. Almost sure nonobservability and unstabilizability of unstable noisy linear plants via communication channel with packet losses. In *Proceedings of the 44th IEEE Conference on Decission and Control and European Control Conference*, pages 7338–7341, Sevilla, Spain, 2005.

[38] Y. Minami, S. Azuma, and T. Sugie. An optimal dynamic quantizer for feedback control with discrete-valued signal constraints. In *Proceedings of the 46th IEEE Conference on Decision and Control*, New Orleans, USA, 2007.

[39] M. Morari and E. Zafiriou. *Robust process control.* Prentice Hall, 1989.

[40] G. Nair and R. Evans. Stabilizability of stochastic linear systems with finite feedback data rates. *SIAM Journal on Control and Optimization*, 43(2):413–436, July 2004.

[41] G. Nair, R. Evans, I. Mareels, and W. Moran. Topological feedback entropy and nonlinear stabilization. *IEEE Transactions on Automatic Control*, 49(9):1585–1597, September 2004.

[42] G. Nair, F. Fagnani, S. Zampieri, and R. Evans. Feedback control under data rate constraints: An overview. *Proceedings of the IEEE*, 95(1):108–137, January 2007.

[43] J. Nilsson. *Real-time Control Systems with Delays.* PhD thesis, Lund Institute of Technology, Sweden, 1998.

[44] S. Norsworthy, R. Schreier, and G. Temes (Eds.). *Delta–Sigma Data Converters: Theory, Design and Simulation.* IEEE Press, Piscataway, NJ, 1997.

[45] D. Něsić and A. Teel. Input-output stability properties of networked control systems. *IEEE Transactions on Automatic Control*, 49(10):1650–1667, 2004.

[46] C. Oară. Constructive solutions to spectral and inner-outer factorizations with respect to the disk. *Automatica*, 41:1855–1866, 2005.

[47] D.E. Quevedo, G.C. Goodwin, and J.A. De Doná. Finite constraint set receding horizon control. *International Journal of Robust and Nonlinear Control*, 14:355–377, March 2004.

[48] A.V. Savkin. Analysis and synthesis of networked control systems: Topological entropy, observability, robustness and optimal control. *Automatica*, 42:51–62, 2006.

[49] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S.S. Sastry. Foundations of control and estimation over lossy networks. *Proceedings of the IEEE*, 95(1):163 – 187, January 2007.

[50] R. Schreier and G.C. Temes. *Understanding Delta Sigma Data Converters.* Wiley-IEEE Press, 2004.

[51] P. Seiler and R. Sengupta. An $\mathcal{H}_\infty$ approach to networked control. *IEEE Transactions on Automatic Control*, 50(3), 2005.

[52] E.I. Silva, G.C. Goodwin, and D.E. Quevedo. On networked control architectures for MIMO plants. In *Proceedings of the 17th IFAC World Congress*, Seoul, Korea, 2008.

[53] E.I. Silva, G.C. Goodwin, D.E. Quevedo, and M.S. Derpich. Optimal noise shaping for networked control systems. In *Proceedings of the European Control Conference*, Kos, Greece, 2007.

[54] E.I. Silva, D.E. Quevedo, and G.C. Goodwin. Optimal coding for bit-rate limited networked control systems in the presence of data loss. In *Proc. of the 46th IEEE Conference on Decision and Control*, New Orleans, USA, 2007.

[55] S. Skogestad and I. Postlethwaite. *Multivariable Feedback Control: Analysis and Design.* Wiley, New York, 1996.

[56] T. Söderström. *Discrete-time stochastic systems.* Prentice Hall, 1994.

[57] H. Spang and P. Schultheiss. Reduction of quantizing noise by use of feedback. *IRE Trans. Comm. Syst.*, CS-10(4):373–380, December 1962.

[58] P.L. Tang and C.W. de Silva. Compensation for transmission delays in an ethernet-based control network using variable-horizon predictive control. *IEEE Transactions on Control and Systems Technology*, 14(4):707–718, July 2006.

[59] S. Tatikonda and S. Mitter. Control over noisy channels. *IEEE Transactions on Automatic Control*, 49(7):1196–1201, July 2004.

[60] S. Tatikonda and S. Mitter. Control under communication constraints. *IEEE Transactions on Automatic Control*, 49(7):1056–1068, July 2004.

[61] S. Tatikonda, A. Sahai, and S. Mitter. Stochastic linear control over a communication channel. *IEEE Transactions on Automatic Control*, 49(9):1549–1561, September 2004.

[62] Y. Tipsuwan and M.Y. Chow. Control methodologies in networked control. *Control Engineering Practice*, 11:1099–1111, 2003.

[63] J. Tuqan and P.P. Vaidyanathan. Statistically optimum pre- and postfiltering in quantization. *IEEE Transactions on Circuits and Systems II*, 44(1):1015–1031, 1997.

[64] M. Vidyasagar. *Control systems synthesis: A Factorization Approach.* MIT Press, Cambridge, USA, 1985.

[65] R.A. Wannamaker, S.P. Lipshitz, J. Vanderkooy, and J. N. Wright. A theory of non-subtractive dither. *IEEE Transactions on Signal Processing*, 48(2):499–516, February 2000.

[66] L. Xiao, M. Johansson, H. Hindi, S. Boyd, and A. Goldsmith. Joint optimization of communication rates and linear systems. *IEEE Transactions on Automatic Control*, 48(1):148–153, January 2003.

[67] K. Zhou, J.C. Doyle, and K. Glover. *Robust and optimal control.* Prentice Hall, Englewood Cliffs, N.J., 1996.