Analysis of the Glottal Pressures

Luis A. Mora [*,**]

[*] *Department of Electronic Engineering, Universidad Técnica Federico Santa María, Chile (e-mail:luis.moraa@sansano.usm.cl).*
[**] *Laboratory of Instrumentation, Control and Automation, Universidad Nacional Experimental del Táchira, Venezuela.*

Abstract: This report complements the paper *A port-Hamiltonian Fluid-Structure Interaction Model for the Vocal Folds* (Mora et al., 2017), submitted to 6th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control (LHMNC18).

In this report a quasi-steady analysis of the glottal pressures is presented. In order that the model proposed in Mora et al. (2017) describes the behavior of the vocal folds, it is necessary that the pressures P_a and P_c correspond to the pressures in the glottis. To obtain an expression for P_a and P_c, we consider an expansion of the control volume studied in Mora et al. (2017) to include the subglottal and epiglottal sections (see Figure 1).

1. SUBGLOTTAL AND EPIGLOTTAL PRESSURES

The pressures in the subglottal and epiglottal sections have been studied, by example, by Ishizaka and Flanagan (1972), Matsudaira and Ishizaka (1972), and Story and Titze (1995). From Ishizaka and Flanagan (1972), the pressure P_c in the epiglottis is given by

$$P_c = P_i - k_e \frac{\rho}{2} \left(\frac{Q_c}{A_c} \right)^2 \tag{1}$$

where P_i is the supraglottal pressure, A_c is the cross-sectional area in the point c, Q_c is the horizontal flow through A_c and

$$k_e = 2 \frac{A_c}{A_i} \left(1 - \frac{A_c}{A_i} \right)$$

is the epiglottal pressure recovery coefficient (Story and Titze, 1995), and A_i is the epiglottis transverse area.

Similarly, from Ishizaka and Flanagan (1972), the pressure P_a in the subglottis is given by:

$$P_a = P_s - \frac{\rho}{2} Q_a^2 \left(\frac{k_s}{A_a} - \frac{1}{A_s} \right) \tag{2}$$

where P_s is the subglottal pressure, A_s is the subglottis cross-sectional area, Q_a is the horizontal flow through the cross-sectional area A_a, $k_s = 1 + \lambda_s$ and λ_s is a loss factor due to the abrupt contraction in the cross-sectional area inlet to the glottis ($\lambda_s = 0.37$).

2. PRESSURES IN THE CONTROL VOLUME

In this section we compute the pressures in Ω_1 and Ω_2 in Figure 1. In Mora et al. (2017), a bi-dimensional airflow between points a and c is considered, i.e., the air velocity vector is defined as $\mathbf{v} = [\vartheta \ \varphi]^T$. The horizontal and vertical air velocities in Ω_1 are given by

$$\vartheta_1 = \begin{cases} \theta_1 - \dfrac{\psi_1}{h_1} x & y \le h_b \\ \dfrac{\psi_1}{h_1} (\ell_0 - x) & y > h_b \end{cases} \tag{3}$$

$$\varphi_1 = \frac{\psi_1}{h_1} y \tag{4}$$

The flows through surfaces Ξ_a and Ξ_b are given by

$$\begin{aligned} Q_a &= A_a \langle \vartheta_1 \rangle_{\Xi_a} \\ &= A_a \left(\theta_1 \frac{h_b}{h_1} + \psi_1 \frac{2\ell_0}{h_1} \left(1 - \frac{1}{2} \frac{h_b}{h_1} \right) \right) \end{aligned} \tag{5}$$

$$\begin{aligned} Q_b &= A_b \langle \vartheta_1 \rangle_{\Xi_b} \\ &= A_b \left(\theta_1 - \psi_1 \frac{\ell_0}{h_1} \right) \end{aligned} \tag{6}$$

Remark 1. Considering that $A_1 = 2\ell_0 L$, $A_a = h_1 L$ and $A_b = h_b L$, θ_1 can be rewritten as

$$\theta_1 = A_b^{-1} \left(Q_b + \frac{1}{2} A_1 \psi_1 \right) \tag{7}$$

and

$$\theta_1 \frac{h_b}{h_1} = A_a^{-1} \left(Q_b + \frac{1}{2} A_1 \psi_1 \right) \tag{8}$$

Then, the relationship between the flows in Ξ_a and Ξ_b is given by

$$Q_a = Q_b + A_1 \psi_1 \tag{9}$$

□

Substituting (9) in (2), P_a can be rewritten as

$$\begin{aligned} P_a = P_s &+ \frac{\rho}{2} Q_b^2 \left(A_s^{-2} - k_s A_a^{-2} \right) \\ &+ \frac{\rho}{2} \left(A_s^{-2} - k_s A_a^{-2} \right) \left(2 Q_b A_1 \psi_1 + A_1^2 \psi_1^2 \right) \end{aligned} \tag{10}$$

Additionally, for Ω_2 the air velocities are given by

$$\vartheta_2 = \begin{cases} \theta_2 - \dfrac{\psi_2}{h_2} x & y \le h_b \\ -\dfrac{\psi_2}{h_2} (\ell_0 + x) & y > h_b \end{cases} \tag{11}$$

$$\varphi_2 = \frac{\psi_2}{h_2} y \tag{12}$$

The flows through surfaces Ξ_c and Ξ_b are given by

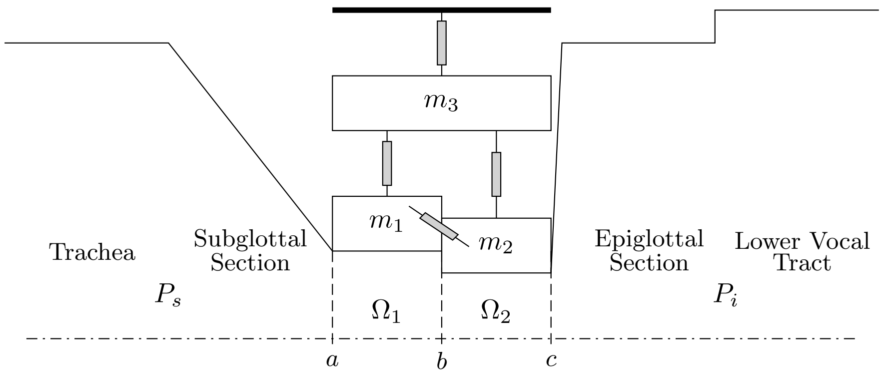

Fig. 1. Glottis structure

$$
\begin{aligned}
Q_c &= A_c \left\langle \vartheta_2 \right\rangle_{\Xi_c} \\
&= A_c \left(\theta_2 \frac{h_b}{h_2} - \psi_2 \frac{2\ell_0}{h_2} \left(1 - \frac{1}{2}\frac{h_b}{h_2} \right) \right) \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
Q_b &= A_b \left\langle \vartheta_2 \right\rangle_{\Xi_b} \\
&= A_b \left(\theta_2 + \psi_2 \frac{\ell_0}{h_2} \right) \qquad (14)
\end{aligned}
$$

Remark 2. Considering that $A_2 = 2\ell_0 L$, $A_c = h_2 L$ and $A_b = h_b L$, θ_2 can be rewritten as

$$
\theta_2 = A_b^{-1} \left(Q_b - \frac{1}{2} A_2 \psi_2 \right) \qquad (15)
$$

and

$$
\theta_2 \frac{h_b}{h_2} = A_c^{-1} \left(Q_b - \frac{1}{2} A_2 \psi_2 \right) \qquad (16)
$$

Then, the relationship between the flows in Ξ_a and Ξ_b is given by

$$
Q_c = Q_b - A_2 \psi_2 \qquad (17)
$$

□

Then, P_c can be rewritten as

$$
\begin{aligned}
P_c &= P_i - \frac{\rho}{2} Q_b^2 k_e A_c^{-2} \\
&\quad - \frac{\rho}{2} k_e A_c^{-2} \left(2 Q_b A_2 \psi_2 - A_2^2 \psi_2^2 \right) \qquad (18)
\end{aligned}
$$

Also, we analyze the pressure in $\{\Omega_1, \Omega_2\}$ using the unsteady Bernoulli equation (Bird et al., 2014), i.e.,

$$
\frac{d\mathbf{v}}{dt} + \frac{1}{2} \nabla |\mathbf{v}|^2 + \frac{1}{\rho} \nabla P = 0 \qquad (19)
$$

The solution to (19) for Ω_1 is given by

$$
P_1 = \begin{cases}
p_{10} + \frac{\rho}{2} \left[\theta_1^2 - \left(\theta_1 - \frac{\psi_1}{h_1} x \right)^2 \right. \\
\left. - \left(\frac{\psi_1}{h_1} y \right)^2 - \left(2\dot{\theta}_1 x + C_1 (y^2 - x^2) \right) \right], & y \le h_b \\
p_{10} + \frac{\rho}{2} \left[\left(\frac{\psi_1}{h_1} \ell_0 \right)^2 - \left(\frac{\psi_1}{h_1} (\ell_0 - x) \right)^2 \right. \\
\left. - \left(\frac{\psi_1}{h_1} y \right)^2 - C_1 (y^2 + 2\ell_0 x - x^2) \right], & y > h_b
\end{cases} \qquad (20)
$$

where p_{10} is the static pressure at the point $(x = 0, y = 0) \in \Omega_1$ and $C_1 = \frac{\dot{\psi}_1}{h_1} - \left(\frac{\psi_1}{h_1} \right)^2$.

Similarly, the pressure in Ω_2 is given by

$$
P_2 = \begin{cases}
p_{20} + \frac{\rho}{2} \left[\theta_2^2 - \left(\theta_2 - \frac{\psi_2}{h_2} x \right)^2 \right. \\
\left. - \left(\frac{\psi_2}{h_2} y \right)^2 - \left(2\dot{\theta}_2 x + C_2 (y^2 - x^2) \right) \right], & y < h_b \\
p_{20} + \frac{\rho}{2} \left[\left(\frac{\psi_2}{h_2} \ell_0 \right)^2 - \left(\frac{\psi_2}{h_2} (\ell_0 + x) \right)^2 \right. \\
\left. - \left(\frac{\psi_2}{h_2} y \right)^2 - C_2 (y^2 - 2\ell_0 x - x^2) \right], & y \le h_b
\end{cases} \qquad (21)
$$

where p_{20} is the static pressure at the point $(x = 0, y = 0) \in \Omega_2$ and $C_2 = \frac{\dot{\psi}_2}{h_2} - \left(\frac{\psi_2}{h_2} \right)^2$.

3. MIDPOINT PRESSURE

In this section we compute a the pressure in midpoint b from Ω_1 and Ω_2, and obtain a relationship between $\frac{\rho}{2} Q_b^2$ and the supraglottal and subglottal pressures. This is necessary to express P_a and P_c as a function of P_s and P_i.

From (20) we have that the pressures in a and b points are given by

$$
\begin{aligned}
P_a &= \left\langle P_1 \right\rangle_{\Xi_a} \\
&= p_{10} - \frac{\rho}{2} \frac{Q_a^2}{A_a^2} + \frac{\rho}{2} \frac{h_b^2}{h_1^2} \theta_1^2 + \rho \theta_1 \psi_1 \frac{h_b \ell_0}{h_1^2} \left(1 - \frac{h_b}{h_1} \right) \\
&\quad + \frac{\rho}{2} \psi_1^2 \left(-\frac{1}{3} + \frac{\ell_0^2}{h_1^2} \left(1 - \frac{h_b}{h_1} \right)^2 \right) + \rho \dot{\theta}_1 \frac{h_b \ell_0}{h_1} \\
&\quad - \frac{\rho}{2} C_1 \left(\frac{h_1^2}{3} - 3\ell_0^2 + 2 \frac{h_b \ell_0^2}{h_1} \right) \qquad (22)
\end{aligned}
$$

$$
\begin{aligned}
P_b &= \left\langle P_1 \right\rangle_{\Xi_b} \\
&= p_{10} - \frac{\rho}{2} \frac{Q_b^2}{A_b^2} + \frac{\rho}{2} \theta_1^2 - \frac{\rho}{2} \frac{h_b^2}{h_1^2} \frac{\psi_1^2}{3} - \rho \dot{\theta}_1 \ell_0 \\
&\quad - \frac{\rho}{2} C_1 \left(\frac{h_b^2}{3} - \ell_0^2 \right) \qquad (23)
\end{aligned}
$$

Equating (10) and (22), solving for p_{10}, and substituting in (23), we obtain

$$
\begin{aligned}
P_b = P_s &+ \frac{\rho}{2}Q_b^2\left(\frac{1-k_s}{A_a^2} - \frac{1}{A_b^2} + \frac{1}{A_s^2}\right) - \dot{\theta}_1\rho\ell_0\left(1+\frac{h_b}{h_1}\right) \\
&+ \frac{\rho}{2}\left(\frac{1-k_s}{A_a^2} + \frac{1}{A_s^2}\right)\left(2Q_bA_1\psi_1 + A_1^2\psi_1^2\right) \\
&- \frac{\rho}{2}\left(1-\frac{h_b}{h_1}\right)\left(\psi_1^2\frac{\ell_0^2}{h_1^2}\left(1-\frac{h_b}{h_1}\right) + 2\psi_1\theta_1\frac{\ell_0 h_b}{h_1^2}\right. \\
&\left.+2C_1\ell_0^2\right) + \frac{\rho}{2}\left(1-\frac{h_b^2}{h_1^2}\right)\left(\theta_1^2 + \frac{\psi_1}{3} + \frac{C_1h_1^2}{3}\right) \qquad (24)
\end{aligned}
$$

which is the pressure in the midpoint b from Ω_1.

Furthermore, from (21), the pressures at points c and b are given by

$$
\begin{aligned}
P_c &= \langle P_2\rangle_{\Xi_c} \\
&= p_{20} - \frac{\rho}{2}\frac{Q_c^2}{A_c^2} + \frac{\rho}{2}\frac{h_b^2}{h_2^2}\theta_2^2 - \rho\theta_2\psi_2\frac{h_b\ell_0}{h_2^2}\left(1-\frac{h_b}{h_2}\right) \\
&- \frac{\rho}{2}\psi_2^2\left(\frac{1}{3} - \frac{\ell_0^2}{h_2^2}\left(1-\frac{h_b}{h_2}\right)^2\right) - \rho\dot{\theta}_2\frac{h_b\ell_0}{h_2} \\
&- \frac{\rho}{2}C_2\left(\frac{h_2^2}{3} - 3\ell_0^2 + 2\frac{h_b\ell_0^2}{h_2}\right) \qquad (25) \\
P_b &= \langle P_2\rangle_{\Xi_b} \\
&= p_{20} - \frac{\rho}{2}\frac{Q_b^2}{A_b^2} + \frac{\rho}{2}\theta_2^2 - \frac{\rho}{2}\frac{h_b^2}{h_2^2}\frac{\psi_2^2}{3} + \rho\dot{\theta}_2\ell_0 \\
&- \frac{\rho}{2}C_2\left(\frac{h_b^2}{3} - \ell_0^2\right) \qquad (26)
\end{aligned}
$$

Equating (18) and (25), solving for p_{20}, and substituting in (26), we obtain that

$$
\begin{aligned}
P_b = P_i &+ \frac{\rho}{2}Qb^2\left(\frac{1-k_e}{A_c^2} - \frac{1}{A_b^2}\right) + \dot{\theta}_2\rho\ell_0\left(1+\frac{h_b}{h_2}\right) \\
&+ \frac{\rho}{2}\left(1-\frac{h_b^2}{h_2^2}\right)\left(\theta_2^2 + \frac{\psi_2}{3} + \frac{C_2h_2^2}{3}\right) \\
&- \frac{\rho}{2}\left(1-\frac{h_b}{h_2}\right)\left(\psi_2^2\frac{\ell_0^2}{h_2^2}\left(1-\frac{h_b}{h_2}\right) - 2\psi_2\theta_2\frac{\ell_0h_b}{h_2^2}\right. \\
&\left.+2C_2\ell_0^2\right) - \frac{\rho}{2}\frac{1-k_e}{A_c^2}\left(2Q_bA_2\psi_2 - A_2^2\psi_2^2\right) \qquad (27)
\end{aligned}
$$

which is the pressure in the midpoint b from Ω_2.

Equating (24) and (27), and solving for $\frac{\rho}{2}Q_b^2$, we obtain

$$
\frac{\rho}{2}Q_b^2\left(\frac{1-k_e}{A_c^2} - \frac{1}{A_s^2} - \frac{1-k_s}{A_a^2}\right) = (P_s - P_i) + f_{b1} + f_{b2}^* \qquad (28)
$$

where

$$
\begin{aligned}
f_{b1} &= \frac{\rho}{2}\left(A_s^{-2} + (1-k_s)A_a^{-2}\right)\left(2Q_bA_1\psi_1 + A_1^2\psi_1^2\right) \\
&+ \frac{\rho}{2}\left(1-\frac{h_b^2}{h_1^2}\right)\left(\theta_1^2 + \frac{\psi_1^2}{3} + \frac{C_1h_1^2}{3}\right) \\
&- \frac{\rho}{2}\left(1-\frac{h_b}{h_1}\right)\left(\psi_1^2\frac{\ell_0^2}{h_1^2}\left(1-\frac{h_b}{h_1}\right) + 2\psi_1\theta_1\frac{\ell_0h_b}{h_1^2}\right. \\
&\left.+2C_1\ell_0^2\right) - \dot{\theta}_1\ell_0\rho\left(1+\frac{h_b}{h_1}\right) \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
f_{b2}^* &= \frac{\rho}{2}A_c^{-2}(1-k_e)\left(2Q_bA_2\psi_2 - A_2^2\psi_2^2\right) \\
&- \frac{\rho}{2}\left(1-\frac{h_b^2}{h_2^2}\right)\left(\theta_2^2 + \frac{\psi_2^2}{3} + \frac{C_2h_2^2}{3}\right) \\
&+ \frac{\rho}{2}\left(1-\frac{h_b}{h_2}\right)\left(\psi_2^2\frac{\ell_0^2}{h_2^2}\left(1-\frac{h_b}{h_2}\right) - 2\psi_2\theta_2\frac{\ell_0h_b}{h_2^2}\right. \\
&\left.+2C_2\ell_0^2\right) - \dot{\theta}_2\ell_0\rho\left(1+\frac{h_b}{h_2}\right) \qquad (30)
\end{aligned}
$$

Remark 3. Notice that when $h_b = h_1$, the second and third terms on the right-hand side of (29) are equal to zero. □

Considering that

$$
\frac{\rho}{2}(1-k_e)\frac{Q_b^2}{A_c^2} = \frac{\rho}{2}(1-k_e)\left(\frac{Q_b^2}{A_b^2} - \left(1-\frac{h_b^2}{h_2^2}\right)\frac{Q_b^2}{A_b^2}\right) \qquad (31)
$$

the equation (28) can be rewritten as

$$
\frac{\rho}{2}Q_b^2\left(\frac{1-k_e}{A_b^2} - \frac{1}{A_s^2} - \frac{1-k_s}{A_a^2}\right) = (P_s - P_i) + f_{b1} + f_{b2} \qquad (32)
$$

where

$$
\begin{aligned}
f_{b2} &= \frac{\rho}{2}A_c^{-2}(1-k_e)\left(2Q_bA_2\psi_2 - A_2^2\psi_2^2\right) \\
&- \frac{\rho}{2}\left(1-\frac{h_b^2}{h_2^2}\right)\left(\theta_2^2 - (1-k_e)\frac{Q_b^2}{A_b^2} + \frac{\psi_2^2}{3} + \frac{C_2h_2^2}{3}\right) \\
&+ \frac{\rho}{2}\left(1-\frac{h_b}{h_2}\right)\left(\psi_2^2\frac{\ell_0^2}{h_2^2}\left(1-\frac{h_b}{h_2}\right) - 2\psi_2\theta_2\frac{\ell_0h_b}{h_2^2}\right. \\
&\left.+2C_2\ell_0^2\right) - \dot{\theta}_2\ell_0\rho\left(1+\frac{h_b}{h_2}\right) \qquad (33)
\end{aligned}
$$

Remark 4. Notice that when $h_b = h_2$, the second and third terms on the right-hand side of (33) are equal to zero. □

Defining f_a as

$$
f_a = \frac{\rho}{2}\left(2Q_bA_1\psi_1 + A_1^2\psi_1^2\right) \qquad (34)
$$

and from Remark 1, we obtain

$$
\begin{aligned}
\theta_1^2 &= \left[A_b^{-1}Q_b + \frac{1}{2}\frac{A_1}{A_a}\psi_1\right]^2 \\
&= \left[A_b^{-1}\left(Q_b + \frac{1}{2}A_1\psi_1\frac{h_b}{h_1}\right)\right]^2 \\
&= A_b^{-2}\left[\left(Q_b + \frac{1}{2}A_1\psi_1\right) - \frac{1}{2}A_1\psi_1\left(1-\frac{h_b}{h_1}\right)\right]^2 \\
&= \frac{1}{2}A_b^{-2}\left(2Q_bA_1\psi_1 + A_1^2\psi_1^2\right)\left(1-\frac{h_b}{h_1}\right) \\
&+ \frac{1}{4}A_b^{-2}\left[\left(2Q_a - A_1\psi_1\right)^2 + A_1^2\psi_1^2\left(1-\frac{h_b}{h_1}\right)^2\right] \\
&= \frac{A_b^{-2}}{\rho}f_a + \frac{A_b^{-2}}{4}\left(2Q_a - A_1\psi_1\right)^2 \\
&+ \frac{A_b^{-2}}{4}A_1^2\psi_1^2\left(1-\frac{h_b}{h_1}\right)^2 \qquad (35)
\end{aligned}
$$

$$F_1^* = \psi_1^2 \frac{\ell_0^2}{h_1^2}\left(1-\frac{h_b}{h_1}\right)^2 + 2\psi_1\theta_1\frac{\ell_0 h_b}{h_1^2}\left(1-\frac{h_b}{h_1}\right)$$
$$= \left(\psi_1\frac{\ell_0}{h_1}\left(1-\frac{h_b}{h_1}\right)+\theta_1\frac{h_b}{h_1}\right)^2 - \theta_1^2\frac{h_b^2}{h_1^2}$$
$$= A_a^{-2}\left(Q_b+\frac{1}{2}A_1\psi_1\right)^2 - \theta_1^2\frac{h_b^2}{h_1^2}$$
$$= A_a^{-2}\left[\left(Q_b+\frac{1}{2}A_1\psi_1\right)^2 - \left(Q_b-\frac{1}{2}A_1\psi_1\frac{h_b}{h_1}\right)^2\right]$$
$$= \frac{A_a^{-2}}{2}\left(2Q_bA_1\psi_1 + A_1^2\psi_1^2\right)\left(1-\frac{h_b}{h_1}\right)$$
$$- \frac{A_a^{-2}}{4}A_1^2\psi_1^2\left(1-\frac{h_b}{h_1}\right)^2$$
$$= \left(1-\frac{h_b}{h_1}\right)\left[\frac{A_a^{-2}}{\rho}f_a - \frac{A_a^{-2}}{4}A_1^2\psi_1^2\left(1-\frac{h_b}{h_1}\right)\right] \quad (36)$$

and

$$\frac{\psi_1^2}{3} + \frac{C_1h_1^2}{3} = \frac{\dot{\psi}_1 h_1}{3} \quad (37)$$

Substituting (34) in (29), we can rewrite f_{b1} as

$$f_{b1} = f_a\left[\frac{1}{A_s^2} + \frac{1-k_s}{A_a^2}\right] - \dot{\theta}_1\ell_0\rho\left(1+\frac{h_b}{h_1}\right) + f_1 \quad (38)$$

where f_1 is formed by the second and third terms on the right-hand side of equation (29). From Remark 3 and using (35), (36) and (37), we obtain that f_1 is equal to 0 when $h_b = h_1$, and when $h_b = h_2$ is given by

$$f_1 = \frac{\rho}{2}\left(1-\frac{h_2^2}{h_1^2}\right)\left(\frac{A_c^{-2}}{\rho}f_a\left(1-\frac{h_2}{h_1}\right) + \frac{\dot{\psi}_1h_1}{3}\right.$$
$$\left. + \frac{A_c^{-2}}{4}\left(2Q_a - A_1\psi_1\right)^2 + \frac{A_c^{-2}}{4}A_1^2\psi_1^2\left(1-\frac{h_2}{h_1}\right)^2\right)$$
$$- \frac{\rho}{2}\left(1-\frac{h_2}{h_1}\right)\left(\frac{A_a^{-2}}{\rho}f_a - \frac{A_1^2}{4A_a^2}\psi_1^2\left(1-\frac{h_2}{h_1}\right) + 2C_1\ell_0^2\right)$$

Similarly, defining f_c as

$$f_c = \frac{\rho}{2}\left(2Q_bA_2\psi_2 - A_2^2\psi_2^2\right) \quad (39)$$
$$= \frac{\rho}{2}\left(2Q_cA_2\psi_2 + A_2^2\psi_2^2\right) \quad (40)$$

and, from Remark 2, we obtain

$$F_2^* = \theta_2^2 - \frac{1-k_e}{A_b^2}Q_b^2$$
$$= k_e\frac{Q_b^2}{A_b^2} + \frac{A_2\psi_2h_b}{4A_b^2h_2^2}\left(4Q_bh_2 + A_2\psi_2h_b\right)$$
$$= k_e\frac{Q_b^2}{A_b^2} + \frac{1}{A_b^2}\frac{h_b}{h_2}\left(Q_bA_2\psi_2 + \frac{A_2\psi_2}{4}\frac{h_b}{h_2}\right)$$
$$= \frac{A_b^{-2}}{2}\frac{h_b}{h_2}\left(2Q_bA_2\psi_2 - A_2^2\psi_2^2\right)$$
$$+ \frac{A_b^{-2}}{2}\frac{h_b}{h_2}A_2^2\psi_2^2\left(1+\frac{1}{2}\frac{h_b}{h_2}\right) + k_eA_b^{-2}(Q_c + A_2\psi_2)^2$$
$$= \frac{A_b^{-2}}{\rho}f_c\left(2k_e + \frac{h_b}{h_2}\right) + k_eA_b^{-2}Q_c^2$$
$$+ \frac{A_b^{-2}}{2}\frac{h_b}{h_2}A_2^2\psi_2^2\left(1+\frac{1}{2}\frac{h_b}{h_2}\right) \quad (41)$$

$$F_3^* = -\left[\psi_2^2\frac{\ell_0^2}{h_2^2}\left(1-\frac{h_b}{h_2}\right)^2 + 2\psi_2\theta_2\frac{\ell_0h_b}{h_2^2}\left(1-\frac{h_b}{h_2}\right)\right]$$
$$= \left(1-\frac{h_b}{h_2}\right)\left[\frac{A_c^{-2}}{\rho}f_c + \frac{A_c^{-2}}{4}A_2^2\psi_2^2\left(1-\frac{h_b}{h_2}\right)\right] \quad (42)$$

and

$$\frac{\psi_2^2}{3} + \frac{C_2h_2^2}{3} = \frac{\dot{\psi}_2h_2}{3} \quad (43)$$

Substituting (39) in (33), we can rewrite f_{b2} as

$$f_{b2} = f_c\left(\frac{1-k_e}{A_c^2}\right) - \dot{\theta}_2\ell_0\rho\left(1+\frac{h_b}{h_2}\right) + f_2 \quad (44)$$

where f_2 is formed by the second and third terms on the right-hand side on equation (33), i.e., from Remark 4 and using (41), (42) and (43), f_2 is equal to 0 when $h_b = h_2$, and when $h_b = h_1$ is given by

$$f_2 = -\frac{\rho}{2}\left(1-\frac{h_1^2}{h_2^2}\right)\left(\frac{A_a^{-2}}{\rho}f_c\left(2k_e+\frac{h_1}{h_2}\right) + \frac{\dot{\psi}_2h_2}{3}\right.$$
$$\left. + k_eA_a^{-2}Q_c^2 + \frac{A_a^{-2}}{2}\frac{h_1}{h_2}A_2^2\psi_2^2\left(1+\frac{1}{2}\frac{h_1}{h_2}\right)\right)$$
$$+ \frac{\rho}{2}\left(1-\frac{h_1}{h_2}\right)\left(2C_2\ell_0^2 - \frac{A_2^2}{4A_c^2}\psi_2^2\left(1-\frac{h_1}{h_2}\right) - \frac{A_c^{-2}}{\rho}f_c\right)$$

Finally, substituting (38) and (44) in (32) we obtain

$$\frac{\rho}{2}Q_b^2 = k_b\left[(P_s - P_i) + f_a\left(\frac{1}{A_s^2} + \frac{(1-k_s)}{A_a^2}\right) + f_c\frac{(1-k_e)}{A_c^2}\right.$$
$$\left. + f_1 + f_2 - \dot{\theta}_1\ell_0\rho\left(1+\frac{h_b}{h_1}\right) - \dot{\theta}_2\ell_0\rho\left(1+\frac{h_b}{h_2}\right)\right] \quad (45)$$

where

$$k_b = \frac{1}{(1-k_e)A_b^{-2} - A_s^{-2} - (1-k_s)A_a^{-2}} \quad (46)$$

which gives an expression for $\frac{\rho}{2}Q_b^2$.

4. PRESSURES P_A AND P_C

In this section we obtain expressions for P_a and P_c, that depend on the differential pressure between the subglottal and epiglottal sections $(P_s - P_i)$ from a quasi-steady pressure analysis. These expressions converge to the formulas presented in previous works, under the corresponding assumptions, as we see also below.

Substituting (45) in (10) and (18) we obtain

$$P_a = P_s + (P_s - P_i)k_{a1} + f_ak_{a1}(1-k_e)A_b^{-2}$$
$$+ k_{a1}\left(f_1 + f_2 - \dot{\theta}_1\ell_0\rho\left(1+\frac{h_b}{h_1}\right) - \dot{\theta}_2\ell_0\rho\left(1+\frac{h_b}{h_2}\right)\right)$$
$$+ f_ck_{a1}(1-k_e)A_c^{-2} \quad (47)$$

$$P_c = P_i - (P_s - P_i)k_{c1} - f_ak_{c1}(A_s^{-2} + (1-k_s)A_a^{-2})$$
$$- k_{c1}\left(f_1 + f_2 - \dot{\theta}_1\ell_0\rho\left(1+\frac{h_b}{h_1}\right) - \dot{\theta}_2\ell_0\rho\left(1+\frac{h_b}{h_2}\right)\right)$$
$$- f_ck_{c2} \quad (48)$$

where

$$k_{a1} = \frac{A_s^{-2} - k_s A_a^{-2}}{(1-k_e)A_b^{-2} - A_s^{-2} - (1-k_s)A_a^{-2}} \quad (49)$$

$$k_{c1} = \frac{k_e A_c^{-2}}{(1-k_e)A_b^{-2} - A_s^{-2} - (1-k_s)A_a^{-2}} \quad (50)$$

$$k_{c2} = \frac{k_e A_c^{-2}\left((1-k_e)(A_c^{-2} + A_b^{-2}) - A_s^{-2} - (1-k_s)A_a^{-2}\right)}{(1-k_e)A_b^{-2} - A_s^{-2} - (1-k_s)A_a^{-2}} \quad (51)$$

Recall that f_1 and f_2 depend on the velocities ψ_i and accelerations $\dot{\psi}_i$ of each mass $m_i, i \in \{1,2\}$.

Usual assumptions in the literature, are the following:

- The vocal folds velocity is negligible compared to the horizontal air velocity (Ishizaka and Flanagan, 1972).
- The dynamics $\dot{\theta}_1$ and $\dot{\theta}_2$ are negligible (quasi-steady analysis).
- $|f_c k_{a1}(1-k_e)A_c^{-2}| \ll |f_a k_{a1}(1-k_e)A_b^{-2}|$
- $|f_a k_{c1}(A_s^{-2} + (1-k_s)A_a^{-2})| \ll |f_c k_{c2}|$

then we have that the pressures P_a and P_c can be rewritten as

$$P_a = P_s + (P_s - P_i)k_{a1} + k_{a1}(1-k_e)A_b^{-2} f_a \quad (52)$$
$$P_c = P_i - (P_s - P_i)k_{c1} - f_c k_{c2} \quad (53)$$

Moreover, if we further assume that

- The horizontal flow is uniform through the glottis, i.e., $f_a = 0$ and $f_c = 0$ (Ishizaka and Flanagan, 1972; Story and Titze, 1995).
- The coefficient k_e is negligible, i.e., $k_e \ll 1$ (Story and Titze, 1995).
- The losses in the subglottal section are negligible, i.e., $\lambda_s = 0 \rightarrow k_s = 1$.
- Pressure P_a is negligible when a collision of m_1 occur.
- There is not vocal tract, i.e., $P_i = 0$

then P_a and P_c are given by

$$P_a = P_s \left(1 - s_{\epsilon 1} s_{\epsilon 2} \frac{A_b^2}{A_a^2}\right) s_{\epsilon 1} \quad (54)$$
$$P_c = 0 \quad (55)$$

These expressions are equivalent to the pressure formulas proposed by Steinecke and Herzel (1995).

For the developments presented in Mora et al. (2017), we use the quasi-steady analysis that leads to equations (52) and (53).

Finally, it should be mentioned that the general expressions in (47) and (48) can be used in the model proposed in Mora et al. (2017) with an adjustment of the dissipation factor κ_d and the exchange energy factor κ_x.

REFERENCES

Bird, R.B., Stewart, W.E., Lightfoot, E.N., and Klingenberg, D.J. (2014). *Introductory Transport Phenomena*. Jhon Wiley & Sons, United States of America.

Ishizaka, K. and Flanagan, J.L. (1972). Synthesis of Voiced Sounds From a Two Mass Model of the Vocal Cords. *Bell System Technical Journal*, 51(6), 1233–1268. doi: 10.1002/j.1538-7305.1972.tb02651.x.

Matsudaira, M. and Ishizaka, K. (1972). *Fluid mechanical considerations of vocal cord vibration*. Speech Communications Research Laboratory, Santa Barbara.

Mora, L.A., Yuz, J.I., Ramirez, H., and Gorrec, Y.L. (2017). A port-Hamiltonian Fluid-Structure Interaction Model for the Vocal folds. In *LHMNC18*, submitted.

Steinecke, I. and Herzel, H. (1995). Bifurcations in an asymmetric vocal-fold model. *The Journal of the Acoustical Society of America*, 97(3), 1874–1884. doi: 10.1121/1.412061.

Story, B.H. and Titze, I.R. (1995). Voice simulation with a body cover model of the vocal folds. *The Journal of the Acoustical Society of America*, 97(2), 1249–1260. doi:10.1121/1.412234.