Energy based modelling and control of physical systems Lectures 1 and 2

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References

Relevant references:

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2. A.J. van der Schaft, L2-Gain and Passivity Techniques in Nonlinear Control, Lect. Notes in Control and Information Sciences, Vol. 218, Springer-Verlag, Berlin, 1996, p. 168, 2nd revised and enlarged edition, Springer-Verlag, London, 2000 (Springer Communications and Control Engineering series), p. xvi+249.
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Outline

1. Modelling: what is it and why use it?

2. Dissipative and passive systems

3. Port-Hamiltonian control system

Some basics on modelling

What is a model of a system?

An abstract representation of the reality

An example

Father:	Do your homework
Son:	What if I don't?
Father:	Then you are grounded.

Some basics on modelling

Different kinds of models

Mental: Intuition and experience, verbal: if..., then...

Physical: Scale models, laboratory set-ups

Mathematical: Equations that describe relations between quantities that are important for the behaviour of systems, e.g., laws of nature.

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

Some basics on modelling

***A model depends on the problem context*:**

Simple models

- Linear, small, EDOs..

Bad approximation of the reality

- More complex control/correction

Trade off

Complex models

- Nonlinear, large, PDEs

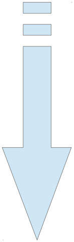
Better approximation of the reality

- Simpler control/correction

Some basics on modelling

What information is used to construct a model?

White Box



Black Box

Based of underlying physics and known parameters

GREY BOX

Based on measured data (I/O signals). No information on the internal structure or relations

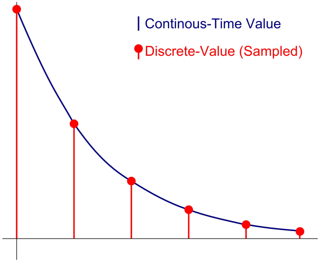
IDENTIFICATION

Some basics on modelling

Different types of mathematical models

Continuous time: $t \in \mathbb{R}$
Differential equations

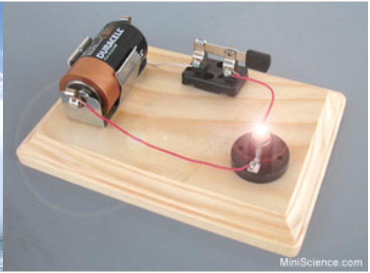
Discrete time: $k \in \mathbb{Z}$
Difference equations



Some basics on modelling

Infinite (distributed parameters)* and *finite (lumped parameters) systems

$$\frac{\partial^2 x}{\partial z^2} = \frac{\partial^2 x}{\partial t^2} \qquad \frac{dx}{dt} = Ax + Bu$$



Modelling

We will consider the following class of systems

- Deterministic finite dimensional (lumped parameters) continuous-time linear or non-linear systems (ODEs).
- Models build using fundamental physical relations, or more precisely *conservation laws*.
- *Open* systems, i.e., systems which interact with the environment through inputs and outputs.

$$\frac{dx(t)}{dt} = \dot{x}(t) = f(x(t), u(t)),$$
$$y(t) = h(x(t), u(t)),$$

with $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. Furthermore $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$.

Such set of ODEs is called a state-space model with state x

Modelling

An ideal control system is composed by:

- Set of **ideal elements** like masses, springs, dampers, tanks, valves, tubes, resistors, capacitors, inductors, diodes, chemical reactants, chemical products, heaters, etc...
- Set of **variables** like velocities, positions, forces, volumes, flows, pressures, voltages, currents, charges, fluxes, mole numbers, chemical potentials, entropy, temperature, etc...
- Set of fundamental **physical relations** like Newton's law, Bernoulli's relations, Maxwell's equations, Gibb's relation, **the first and second principle of Thermodynamics**, etc...
- Set of **interconnection relations** between elements: **Kirchkoff's laws** of current and voltages.

Modelling

We may identify dynamic and static elements:

Dynamic Masses, springs, capacitors, inductors, tanks, etc...

- **Energy conservation**

Static Dampers, transformers, resistors, valves, etc...

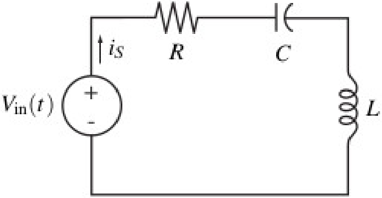
- **Dissipation, scaling $\rightarrow$ non-energy conservative**

There are two kind of fundamental physical relations:

Constitutive: All elements,
Dynamic: Dynamic elements $\Big\} \Rightarrow$ **Balance equations** $\downarrow$ Dynamical system model

Example: RLC circuit

Let us consider a simple linear RLC circuit:



Constitutive relations

$$u_s = V_{in}$$
$$u_r = RI_r$$
$$\phi = LI_L$$
$$Q = Cu_C$$

Dynamic relations

$$u_L = \frac{d\phi}{dt}, \quad \text{or in integral form} \quad \phi(t) = \phi(t_0) + \int_0^t u_L(\tau)d\tau$$

$$I_C = \frac{dQ}{dt}, \quad \text{or in integral form} \quad Q(t) = Q(t_0) + \int_0^t I_C(\tau)d\tau$$

Example: RLC circuit

Interconnection relations (Kirchkoff's laws): $\sum u = 0$ voltage law, $\sum i = 0$ current law

Using the interconnection relations together with the constitutive and dynamical relations we obtain the **state space model**

$$\frac{dQ}{dt} = \frac{\phi}{L}$$

$$\frac{d\phi}{dt} = -\frac{Q}{C} - R\frac{\phi}{L} + V_{in}$$

with state variables $x = [Q, \phi]$ and input V_{in}.

- If the initial conditions $Q(t_0)$ and $\phi(t_0)$ are known, together with the profile V_{in}, then the time evolution of the system is fully determined for all $t > t_0$.

Example: RLC circuit

What about the energy of the systems?

Energy = Energy stored in the capacitor + Energy stored in the inductor

$$H(x(t)) = \frac{1}{2}\frac{\phi^2}{L} + \frac{1}{2}\frac{Q^2}{C}$$

The time variation of the energy is given by

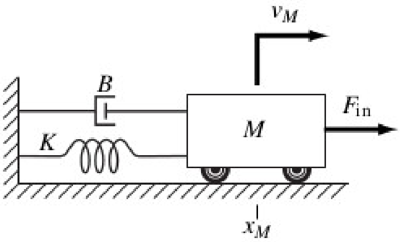
$$\frac{dH(x(t))}{dt} = \frac{\partial H}{\partial x}^\top \frac{dx}{dt} = \left(\frac{Q}{C}\right)\left(\frac{\phi}{L}\right) - \left(\frac{Q}{C}\right)\left(\frac{\phi}{L}\right) + V_{in}\left(\frac{\phi}{L}\right) - R\left(\frac{\phi}{L}\right)^2$$
$$= V_{in}\left(\frac{\phi}{L}\right) - R\left(\frac{\phi}{L}\right)^2 = V_{in}I_L - RI_L^2$$

Hence, the balance equation characterizing the time variation of energy can be written as

$$H(t) = H(t_0) + \int_0^t V_{in}(\tau)I_L(\tau)d\tau - \int_0^t RI_L(\tau)^2 d\tau$$

Example: mass-spring-damper system

Let us consider a simple linear translational MSD system:



Constitutive relations

$$F_s = F_{in}$$
$$F_B = B v_B$$
$$p = M v_M$$
$$q = K^{-1} F_K$$

Dynamic relations

$$F_M = \frac{dp}{dt}, \quad \text{or in integral form} \quad p(t) = p(t_0) + \int_0^t F_M(\tau)d\tau$$

$$v_K = \frac{dq}{dt}, \quad \text{or in integral form} \quad q(t) = q(t_0) + \int_0^t v_K(\tau)d\tau$$

Example: mass-spring-damper system

Using the interconnection relations (Kirchkoff's laws) together with the constitutive and dynamical relations we obtain the **state space model**

$$\frac{dq}{dt} = \frac{p}{M}$$
$$\frac{dp}{dt} = -\frac{q}{K^{-1}} - B\frac{p}{M} + F_{in}$$

with state variables $x = [q, p]$ and input F_{in}.

Example: MSD system

What about the energy of the systems?

Energy = Energy stored in the mass + Energy stored in the spring

$$H(x(t)) = \frac{1}{2}\frac{p^2}{M} + \frac{1}{2}\frac{q^2}{K^{-1}}$$

The time variation of the energy is given by

$$\frac{dH(x(t))}{dt} = \frac{\partial H}{\partial x}^{\top}\frac{dx}{dt} = \left(\frac{q}{K^{-1}}\right)\left(\frac{p}{M}\right) - \left(\frac{q}{K^{-1}}\right)\left(\frac{p}{M}\right) + F_{in}\left(\frac{p}{M}\right) - D\left(\frac{p}{M}\right)^2$$
$$= F_{in}\left(\frac{p}{M}\right) - B\left(\frac{p}{M}\right)^2 = F_{in}v_M - Rv_M^2$$

The balance equation characterizing the time variation of energy can be written as

$$H(t) = H(t_0) + \int_0^t F_{in}(\tau)v_M(\tau)d\tau - \int_0^t Bv_M(\tau)^2 d\tau$$

Port-modelling of physical systems

Let us look closer to the models, and in particular to their balance equations:

$$\text{A component's dynamic relation} \rightarrow x(t) = x(t_0) + \int_0^t u'_{in}(\tau)d\tau$$

And in particular to the energy balance

$$H(t) = H(t_0) + \underbrace{\int_0^t u_{in}(\tau)y(\tau)d\tau}_{\text{supplied energy}} - \underbrace{\int_0^t R(x)y(\tau)^2 d\tau}_{\text{dissipated energy}}$$

The balance equations expresses conservation of some physical quantity: Energy, mass, volume, etc...

The existence of balance equations is the base for dissipative and passive system theory. *All* physical systems are dissipative or passive?

1. Modelling: what is it and why use it?

2. Dissipative and passive systems

3. Port-Hamiltonian control system

Port-modelling of physical systems

Can different domains be approached in a similar way?

- Can they be modelled in a same structured manner?
- Can these models be interconnected in a physical consistent fashion?
- What about the study of solutions and stability properties? Can they be approached using some generalized method?

Most engineering applications are mixtures of different domains. Treating the subsystems related to separate domains differently is time-consuming, and often yields causality issues when interconnecting the subsystems: common problem in signal based modelling. In the nonlinear case the before mentioned questions become critical!

Energy storage, dissipation, and transformation

Properties common to all physical domains

Port-modelling of physical systems

Motivations for adopting an energy-based perspective in modelling

- Physical system can be viewed as a set of simpler subsystems that exchange energy through ports,
- Energy is a concept common to all physical domains and is not restricted to linear or non-linear systems: non-linear approach,
- Energy can serve as a *lingua franca* to facilitate communication among scientists and engineers from different fields,
- Role of energy and the interconnections between subsystems provide the basis for various control techniques: Lyapunov based control.

Dissipative and passive systems

The dynamic behaviour of a physical system is given by sets of *balance equations*. These equations express conservation laws. Conservation of

- Energy
- Mass
- Momentum
- Volume
- etc...

How can we use this for modelling? We need a mathematical system theory to exploit these properties:

Dissipative and passive system theory

Dissipative and passive systems

Consider the system

$$\dot{x}(t) = f(x(t), u(t)), \qquad y(t) = h(x(t), u(t)), \tag{2}$$

with $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. Furthermore $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$. Let us addition define the **supply rate** $w(t) = w(u(t), y(t))$,

$$\int_0^t |w(u(\tau), y(\tau))d\tau| < \infty$$

Dissipative systems

The system (2) is said to be dissipative if there exists a so-called storage function $V(x) \geq 0$ such that the following dissipation inequality holds:

$$V(x(t)) \leq V(x(0)) + \int_0^t w(u(\tau), y(\tau))d\tau$$

along all possible trajectories of (2) starting at $x(0)$, for all $x(0)$, $t \geq 0$.

Dissipative systems

Some comments

- Storage functions are defined up to an additive constant,
- If the system is dissipative with respect to supply rates $w_i(u, y)$, $1 \leq i \leq m$, then the system is also dissipative with respect to any supply rate of the form $\sum_{i=1}^{m} \alpha_i w_i(u, y)$, with $\alpha_i \geq 0$ for all $1 \leq i \leq m$.
- The definition, sometimes referred to as Willems' dissipativity definition, does not require any regularity on the storage functions: it is a very general definition.
- We may find several definitions of dissipativity in the literature

Passive systems

A particular case of dissipative systems are *passive systems*:

Passive systems

Suppose that the system (2) is dissipative with supply rate $w(u, y) = u^T y$ and storage function $V(x(t))$ with $V(0) = 0$; i.e. for all $t \geq 0$ we have that

$$V(x(t)) \leq V(x(0)) + \int_0^t u(\tau)^\top y(\tau) d\tau,$$

Then the system is passive.

Passive systems are a subclass of dissipative systems with the specific properties

- The supply rate is defined by the product between inputs and outputs,
- The storage function is not defined up to a constant ($V(0) = 0$),

Passive systems

The dissipation of a passive system may also be explicitly taken into account:

Strictly passive systems

A system (2) is said to be strictly state passive if it is dissipative with supply rate $w = u^\top y$ and the storage function $V(x(t))$ with $V(0) = 0$, and there exists a positive definite function $S(x)$ such that for all $t \geq 0$:

$$V(x(t)) \leq V(x(0)) + \int_0^t u(\tau)^\top y(\tau) d\tau - \int_0^t S(x(\tau)) d\tau,$$

If the equality holds in the above equation and $S(x) = 0$, then the system is said to be lossless (conservative).

The function $S(x)$ is called the the dissipation rate.

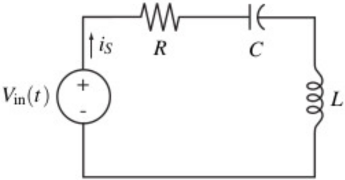
Passive systems

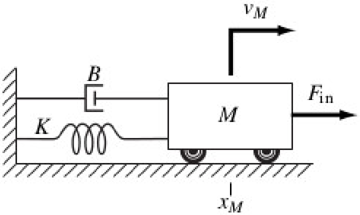
Why are we interested in passive systems?

- Many physical systems are passive with respect to the storage function defined by their physical energy function and with respect to their natural supply rate (given by the physical inputs and outputs),
- Its a non-linear approach (does not require any assumption of linearity),
- The physical energy may be used as a candidate Lyapunov function to analyse stability.
- A “well defined” interconnection of passive system is again a passive system.

Examples: RLC circuit and MSD system

What about our examples? are they passive?





Examples: RLC circuit and MSD system

The RLC circuit

Energy = Energy stored in the capacitor + Energy stored in the inductor

$$H(x(t)) = \frac{1}{2}\frac{\phi^2}{L} + \frac{1}{2}\frac{Q^2}{C}$$

The time variation of the energy is given by

$$\frac{dH(x(t))}{dt} = \frac{\partial H}{\partial x}^\top \frac{dx}{dt} = \left(\frac{Q}{C}\right)\left(\frac{\phi}{L}\right) - \left(\frac{Q}{C}\right)\left(\frac{\phi}{L}\right) + V_{in}\left(\frac{\phi}{L}\right) - R\left(\frac{\phi}{L}\right)^2$$
$$= V_{in}\left(\frac{\phi}{L}\right) - R\left(\frac{\phi}{L}\right)^2 = V_{in}I_L - RI_L^2$$

The time variation of the energy is

$$H(t) = H(t_0) + \int_0^t V_{in}(\tau)I_L(\tau)d\tau - \int_0^t RI_L(\tau)^2 d\tau$$

Examples: RLC circuit and MSD system

$$H(x(t)) = \frac{1}{2}\frac{\phi^2}{L} + \frac{1}{2}\frac{Q^2}{C} \geq 0, \qquad H(0) = 0.$$

Hence H qualifies as a potential storage function. Now,

$$H(t) = H(t_0) + \int_0^t V_{in}(\tau) I_L(\tau) d\tau - \int_0^t R I_L(\tau)^2 d\tau.$$

The system is passive if we choose $u = V_{in}$ and $y = I_L$:

$$H(t) \leq H(t_0) + \int_0^t V_{in}(\tau) I_L(\tau) d\tau.$$

Furthermore, if the we choose the dissipation rate as $S(x) = R I_L(\tau)^2$, then the system is strictly passive

$$H(t) = H(t_0) + \underbrace{\int_0^t u(\tau) y(\tau) d\tau}_{\text{supplied energy}} - \underbrace{\int_0^t S(x(\tau)) d\tau}_{\text{dissipated energy}}.$$

Examples: RLC circuit and MSD system

MSD system

Energy = Energy stored in the mass + Energy stored in the spring

$$H(x(t)) = \frac{1}{2}\frac{p}{M}^2 + \frac{1}{2}\frac{q}{K^{-1}}^2$$

The time variation of the energy is given by

$$\frac{dH(x(t))}{dt} = \frac{\partial H}{\partial x}^\top \frac{dx}{dt} = \left(\frac{q}{K^{-1}}\right)\left(\frac{p}{M}\right) - \left(\frac{q}{K^{-1}}\right)\left(\frac{p}{M}\right) + F_{in}\left(\frac{p}{M}\right) - D\left(\frac{p}{M}\right)^2$$
$$= F_{in}\left(\frac{p}{M}\right) - B\left(\frac{p}{M}\right)^2 = F_{in}v_M - Rv_M^2$$

The balance equation characterizing the time variation of energy can be written as

$$H(t) = H(t_0) + \int_0^t F_{in}(\tau)v_M(\tau)d\tau - \int_0^t Bv_M(\tau)^2 d\tau$$

Examples: RLC circuit and MSD system

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The system is passive if we choose $u = F_{in}$ and $y = v_M$:

$$H(t) \leq H(t_0) + \int_0^t F_{in}(\tau)v_M(\tau)d\tau.$$

Furthermore, if the we choose the dissipation rate as $S(x) = Bv_M(\tau)^2$, then the system is strictly passive

$$H(t) = H(t_0) + \underbrace{\int_0^t u(\tau)y(\tau)d\tau}_{\text{supplied energy}} - \underbrace{\int_0^t S(x(\tau))d\tau}_{\text{dissipated energy}}.$$

Examples: RLC circuit and MSD system

Some remarks

- The chosen inputs and outputs correspond to the physical input and outputs of the system: input voltage and input force / current in the inductor and velocity of the mass
- If we eliminate the resistive components, resistor (R) and damper (B), the supply rate is zero and the system is a lossless (conservative) passive system. Indeed,

$$H(t) = H(t_0) + \underbrace{\int_0^t u(\tau)y(\tau)d\tau}_{\text{supplied energy}}$$

  i.e., the energy is conserved.
- The product $u^\top y$ has the units of **power**, i.e., it defines a power product. This has strong implications for modelling: if the input and outputs define power products the power preserving interconnection of physical (passive) systems defines again a physical (passive) system.

End of the first lesson: Gracias!

Port-Hamiltonian control systems

- We have seen in the first part of this lecture that many physical systems are dissipative or passive. These properties provide a structure for the modelling and analysis of solutions of (non-linear) general control system.
- Question: can we expect an even more specific structure in general control systems?

Port-Hamiltonian control systems

Symplectic techniques in feedback control

- The structure of the dynamical equations may be **related to Mathematical Physics: Lagrangian and Hamiltonian systems** augmented with input-output maps.
- **For mechanical systems, mechanisms and robots**: controlled Lagrangian and Hamiltonian systems (with dissipation).
- **For electrical circuits:** generalized Lagrangian and Hamiltonian systems (with dissipation), dissipative port Hamiltonian systems
- **Network models** of complex and interconnected systems: Port-Hamiltonian systems power conserving interconnections.

Port-Hamiltonian control systems

Each engineering domain consists of two sub-domains:

Electrical	Electrical + Magnetic
Mechanical	Kinetic + Potential
Hydraulic	Hydraulic kinetic + Hydraulic potential

Thermal domain has no sub-domains $\Rightarrow$ **Irreversible creation of entropy**

Port-Hamiltonian control systems

How to treat all domains on equal footing?

The Generalized Bond Graph formalism [Breedveld 1982]

.
The main idea is to decompose the “conventional” engineering domains, i.e., electrical, mechanical and hydraulical into new domains.

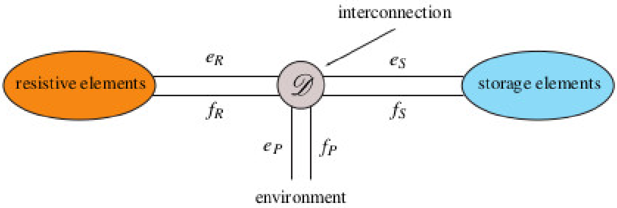
- For each new domain introduce two variables, called power conjugated variables,
- The product of these variables equals power: $V \times I$, $F \times v$, $P \times V$, $T \times S$,*etc*...,
- Label these variables as efforts $e \in \mathbb{E}$ and flows $f \in \mathbb{F}$.

Each element defines a power port, with

$$P = ef$$

Port-Hamiltonian control system

Within this formalism a physical system is defined by the interconnection between energy storage elements, resistive elements, and the environment:



This defines a natural space: $\mathbb{F} := \mathbb{F}_S \times \mathbb{F}_R \times \mathbb{F}_P$; $\mathbb{E} := \mathbb{E}_S \times \mathbb{E}_R \times \mathbb{E}_P$

Port-Hamiltonian control systems

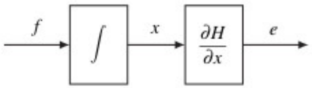
The structure of any energy storing element is the following

$$\dot{x} = f$$

$$x(t) = x(0) + \int_0^t f(\tau)d\tau$$

$$e(t) = \frac{\partial H}{\partial x}(x(t))$$

With $H(x)$ the stored energy of the element. The previous equations can be schematically represented as



Port-Hamiltonian control system

So, we arrive to the following set of variables in the Generalized Bond Graph formalism

physical domain	flow $f \in \mathscr{F}$	effort $e \in \mathscr{F}$	state variable $x = \int f\,dt$
electric	current	voltage	charge
magnetic	voltage	current	flux linkage
potential translation	velocity	force	displacement
kinetic translation	force	velocity	momentum
potential rotation	angular velocity	torque	angular displacement
kinetic rotation	torque	angular velocity	angular momentum
potential hydraulic	volume flow	pressure	volume
kinetic hydraulic	pressure	volume flow	flow tube momentum
chemical	molar flow	chemical potential	number of moles
thermal	entropy flow	temperature	entropy

Jeltsema. D, van Der Schaft, A. Memristive port-Hamiltonian systems, Mathematical and Computer Modelling of Dynamical Systems - MATH COMPUT MODEL DYNAM SYST 01/2010; 16(2). DOI:10.1080/13873951003690824

Port-Hamiltonian control systems

- The constitutive relations are of the form $e = \frac{\partial H}{\partial x}(x(t))$,
- the dynamic relations are of the form $\dot{x} = f$

Furthermore, observe that the change in energy given by $\dot{H} = \frac{dH}{dt}(x(t))$ is now always given by the external power flow of the energy storing element:

$$\dot{H}(x(t)) = \frac{\partial H}{\partial x}(x(t))\dot{x} = e^\top f$$

Hence by construction, the change of energy in time is always the product of flows and efforts

Remark

In a similar manner a resistive element is defined by the static relation

$$e_R = R(f_R), \Rightarrow P_R = e_R f_R = R(f_R) f_R > 0,$$

which defines a positive (dissipative) power product.

Port-Hamiltonian control systems

Dynamic system = System that exchanges Energy

- Storage of energy corresponds to a state.
- The natural physical states are in each engineering domain given by the integrated flow variables x.
- The state variables x are called energy variables, whereas e are the co-energy variables.
- Dynamic system if and only there is exchange of energy among the elements.

Is it possible to generalize the interconnection structure?

Yes! **Dirac structure**

Port-Hamiltonian control systems

The interconnection structure satisfies the power preserving property

$$e_s^\top f_s + e_R^\top f_R + e_p^\top f_p = 0$$

or in terms of the energy storing elements

$$\dot{H}(x(t)) = -e_s^\top f_s = e_R^\top f_R + e_p^\top f_p$$

which yields the energy balance equation

$$H(x(t)) = H(x(0)) + \int_0^t e_R^\top(\tau) f_R(\tau) + e_p^\top(\tau) f_p(\tau) d\tau$$

Dirac structure $\rightarrow$ port-Hamiltonian systems

The power preserving property is defined by the geometric notion of a Dirac structure, which naturally defines port-Hamiltonian control systems.

Port-Hamiltonian control systems

The standard Hamiltonian system is defined by a geometric object (Dirac structure) defined by a Poisson bracket:

$$\{F, G\}(x) = \sum_{k,l=1}^{n} \frac{\partial F}{\partial x_k}(x) J_{kl}(x) \frac{\partial G}{\partial x_l}(x)$$

where

$$J(x) = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \quad x \in \mathbb{R}^{2n},$$

with rank $2n$ everywhere. For any Hamiltonian $H \in C^\infty(\mathbb{R}^{2n})$, the Hamiltonian vector field X_H, is given by the familiar equations of motion:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}(q, p),$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}(q, p), \quad i = 1, \ldots, n$$

called the standard Hamiltonian equations, and $q = (q_1, \ldots, q_n)$ and $p = (p_i, \ldots, p_n)$ are called the generalized configuration coordinates.

Port-Hamiltonian control systems

A major generalization of Hamiltonian systems is to consider systems on a differentiable manifold M with a pseudo-Poisson bracket $\{\cdot,\cdot\}$. Then if one considers local coordinates $x_1, \ldots, x_n$, the port-Hamiltonian system is written as:

$$\dot{x} = J(x)\frac{\partial H}{\partial x} + g(x)u$$
$$y = g(x)^\top \frac{\partial H}{\partial x}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{u} \in \mathbb{R}^m$, $m < n$, is the control action, $H : \mathbb{R}^n \to \mathbb{R}$ is the total stored energy, $J(x) = -J(x)^\top$ is the $n \times n$ natural interconnection matrix, $\mathbf{u}, \mathbf{y} \in \mathbb{R}^m$, are conjugated variables whose product has units of power and $g(x)$, is the $n \times m$ input map. The following energy balance immediately follows

$$\dot{H} = u^\top y$$

showing that a port–Hamiltonian system is a loss-less state space system, and hence a passive system, if the Hamiltonian H is bounded from below.

Port-Hamiltonian control systems

Energy-dissipation is included in the framework of port-Hamiltonian systems by terminating some of the ports by resistive elements. In this case the model is of the form

$$\dot{x} = (J - R)\frac{\partial H}{\partial x} + gu,$$
$$y = g^T \frac{\partial H}{\partial x},$$

where $R(x) = R(x)^\top \geq 0$ is the $n \times n$ damping matrix. In this case the energy-balancing property takes the form

$$\dot{H} = u^\top y - \frac{\partial H}{\partial x}^\top R \frac{\partial H}{\partial x},$$
$$\dot{H} \leq u^\top y,$$

showing that a port-Hamiltonian system is passive if the Hamiltonian H is bounded from below. Note that in this case two geometric structures play a role: the internal interconnection structure given by J, and a dissipative structure given by R.

Port-Hamiltonian control systems

Remarks

- In general, any systems without thermodynamic phenomena can be expressed as PHS.
- Its enough to know the energy function and the interconnection structure (geometry of the system): The dynamic of the system is completely determined by these objects.

Example: the RLC circuit

Let us first consider a lossless LC circuit. The energy is

$$H(x(t)) = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}\frac{\phi^2}{L}$$

The interconnection structure just characterize the exchange of energy between the inductor and the capacitor:

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The internal dynamics of the system is then given by

$$\dot{x} = J\frac{\partial H}{\partial x} = J\begin{bmatrix} \frac{Q}{C} \\ \frac{\phi}{L} \end{bmatrix} = \begin{bmatrix} \frac{\phi}{L} \\ -\frac{Q}{C} \end{bmatrix}$$

Example: the RLC circuit

Let us consider the complete RLC circuit, with dissipation and input port. The energy remains the same

$$H(x(t)) = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}\frac{\phi^2}{L}$$

The interconnection structure just characterize the exchange of energy between the inductor and the capacitor, but in this case we have to add an additional structure matrix that characterizes the dissipation of the system and an input vector field

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix}, \qquad gu = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

The complete dynamics of the system is now given by

$$\dot{x} = (J - R)\frac{\partial H}{\partial x} + gu = (J - R)\begin{bmatrix} \frac{Q}{C} \\ \frac{\phi}{L} \end{bmatrix} + gu = \begin{bmatrix} \frac{\phi}{L} \\ -\frac{Q}{C} - R\frac{\phi}{L} + V_{in} \end{bmatrix}$$

Example: the MSD system

Let us first consider a lossless MS system. The energy is

$$H(x(t)) = \frac{1}{2}\frac{q}{K^{-1}}^{2} + \frac{1}{2}\frac{p}{M}^{2}$$

The interconnection structure just characterize the exchange of energy between the mass and the spring:

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The internal dynamics of the system is then given by

$$\dot{x} = J\frac{\partial H}{\partial x} = J\begin{bmatrix} \frac{q}{K^{-1}} \\ \frac{p}{M} \end{bmatrix} = \begin{bmatrix} \frac{p}{M} \\ -\frac{q}{K^{-1}} \end{bmatrix}$$

Example: the MSD system

Let us consider the complete MSD system, with dissipation and input port. The energy remains the same

$$H(x(t)) = \frac{1}{2}\frac{q}{K^{-1}}^2 + \frac{1}{2}\frac{p}{M}^2$$

The interconnection structure remains the same, but in this case we have to add an additional structure matrix that characterizes the dissipation of the system and an input vector field

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}, \qquad gu = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

The complete dynamics of the system is now given by

$$\dot{x} = (J - R)\frac{\partial H}{\partial x} + gu = (J - R)\begin{bmatrix} \frac{q}{K^{-1}} \\ \frac{p}{M} \end{bmatrix} + gu = \begin{bmatrix} \frac{p}{M} \\ -\frac{q}{K^{-1}} - B\frac{p}{M} + F_{in} \end{bmatrix}$$

Concluding remarks

- Energy based modelling: based on the universal concept of energy transfer.
- Provides physical interpretation to the models and the solutions.
- Passivity is naturally encountered when working with problems arising from physical applications.
- Port-Hamiltonian control systems defines a class of non-linear passive systems which encompasses a large class of physical applications.
- A modelling and control approach which is transversal to different (or combination of) physical domains.