Energy based modelling and control of physical systems Lecture 3

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Summer School on Estimation and Control, Department of Electronic Engineering, Universidad Técnica Federico Santa María, Valparaiso, Chile

Héctor Ramírez Estay

AS2M, FEMTO-ST UMR CNRS 6174,
Université de Franche-Comté,
École Nationale Supérieure de Mécanique et des Microtechniques,
26 chemin de l'épitaphe, F-25030 Besançon, France.

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References

Relevant references:

1. Khalil, H., Nonlinear Systems, Third Edition, Prentice Hall, 2002, ISBN 0-13-067389-7.
2. Brogliato, B., Lozano, R., Maschke, B., and Egeland, O. (2007). Dissipative Systems Analysis and Control. Communications and Control Engineering Series. Springer Verlag, London, 2nd edition edition.
3. A.J. van der Schaft, L2-Gain and Passivity Techniques in Nonlinear Control, Lect. Notes in Control and Information Sciences, Vol. 218, Springer-Verlag, Berlin, 1996, p. 168, 2nd revised and enlarged edition, Springer-Verlag, London, 2000 (Springer Communications and Control Engineering series), p. xvi+249.
4. van der Schaft, A.J and Jeltsema, D. Port-Hamiltonian Systems: from Geometric Network Modeling to Control, Module M13, HYCON-EECI Graduate School on Control, April 07–10, 2009.

Outline

1. Stability: definitions

2. Passivity based control: Damping injection and energy shaping

Some basic notions on stability

We are considering the following class of state space model

$$\begin{aligned}\dot{x}(t) &= f(x(t)),\\ x(0) &= x_0,\end{aligned} \tag{1}$$

with $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Furthermore $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$. It is assumed that $f(x(t))$ satisfies the standard assumptions for existence and uniqueness of solutions, i.e., that $f(x(t))$ is Lipschitz continuous with respect to x, uniformly in t, and piecewise continuous in t.

We want to analyse the dynamics of the system

- Do the solutions of (1) remain bounded in time?
- If yes, do they in addition converge to some *equilibrium point*?
- If yes, can we say anything of the speed of convergence?
- Can we modify the solution with some external control input to impose a desired closed-loop behaviour?

Some basic notions on stability

We are considering the following class of state space model

$$\begin{aligned} \dot{x}(t) &= f(x(t)), \\ x(0) &= x_0, \end{aligned} \tag{2}$$

with $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Furthermore $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. It is assumed that $f(x(t))$ satisfies the standard assumptions for existence and uniqueness of solutions, i.e., that $f(x(t))$ is Lipschitz continuous with respect to x, uniformly in t, and piecewise continuous in t.

Equilibrium position

An equilibrium point x^* is a solution to

$$\dot{x}(x^*) = 0, \qquad \text{i.e.,} \qquad f(x^*) = 0$$

Without loss of generality we will assume that $x^* = 0$.

Some basic notions on stability

An equilibrium position $x = 0$ of system (1) is

1. stable if for any $\epsilon > 0$ and $t_0 \geq 0$, there exists a $\delta(\epsilon, t_0) > 0$ such that $\|x_0\| < \delta$ implies $\|x(t, x_0)\| < \epsilon$ for all $t \geq t_0$,
2. uniformly stable if δ does not depend on t_0,
3. asymptotically stable if it is stable and for any $t_0 \geq 0$ there exists a $\Delta(t_0) > 0$ such that every solution $x(t, x_0)$ of system (1 for which $\|x_0\| < \Delta$ satisfies the relation

$$\lim_{t \to \infty} \|x(t, x_0)\| \to 0 \tag{3}$$

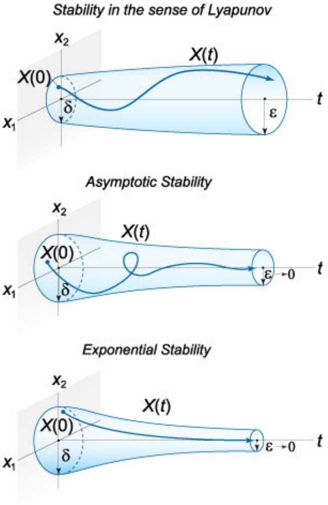
4. uniformly asymptotically stable if it is uniformly stable, Δ does not depend on t_0, and relation (3) holds uniformly with respect to t_0 and x_0 in the domain $t_0 \geq 0$, $\|x_0\| < \Delta$,

Some notions on stability

An equilibrium position $x = 0$ of system (1) is

1. globally asymptotically stable if it is stable and relation (3) holds for any $t_0 \geq 0$ and x_0,
2. uniformly globally asymptotically stable if it is uniformly stable and relation (3) holds for any $t_0 \geq 0$ and x_0 uniformly relative to t_0 and x_0 in the domain $t_0 \geq 0$, $x_0 \in K$, where K is arbitrary compact in the x-space,
3. exponentially asymptotically stable if there exist positive constants Δ, M, and α such that every solution $x(t, x_0)$ of system (1), for which $\|x_0\| < \Delta$, satisfies the relation
$$\|x(t, x0)\| < M\|x_0\|e^{-\alpha(t-t_0)} \tag{4}$$
for all $t \geq t_0 \geq 0$, and
4. globally exponentially asymptotically stable if there exist positive constants M and α such that relation (4) holds for $t \geq t_0 \geq 0$ and arbitrary $t_0 \geq 0$ and x_0.

Some notions on stability



Figures taken from:
http://www.math24.net/

Some notions on stability

How do we analyse stability?

Lyapunov stability theory

- Lyapunov's direct method (second method) $\rightarrow$ non-linear systems
- Lyapunov's indirect method (first method) $\rightarrow$ linear systems

Lyapunov's direct method allows to determine the stability of a system without explicitly integrating the differential equations. The method is a generalization of the idea that if there is some "measure of energy" in a system, then we can study the rate of change of the energy of the system to ascertain stability.

Some notions on stability

Let B_ϵ be a ball of size ϵ around the origin, $B_\epsilon = x \in \mathbb{R}^n : x < \|\epsilon\|$.

Positive definite function

A continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is a locally positive definite function if $V(0) = 0$ and for $x \in B_\epsilon$, $x \neq 0 \to V(x) > 0$. If B_ϵ is the whole state space, then $V(x)$ is globally positive definite.

A positive definite function is like an energy function.

Lyapunov's direct method

Theorem: Lyapunov stability Let $V(x)$ be a non-negative function with continuous partial derivatives such that

- $V(x)$ is positive definite on B_ϵ, and $\dot{V} \leq 0$ locally in x and for all t, then the origin of the system is locally stable (in the sense of Lyapunov).
- If in addition $V(x) \to \infty$ when $\|x\| \to \infty$, then the system is globally stable.

Theorem: Asymptotic stability Let $V(x)$ be a non-negative function with continuous partial derivatives such that

- $V(x)$ is positive definite on B_ϵ, and $\dot{V} < 0$, $\forall x \in B_\epsilon / \{0\}$ and $V(0) = 0$ locally in x and for all t, and then the origin of the system is locally asymptotically stable.
- If in addition $V(x) \to \infty$ when $\|x\| \to \infty$, then the system is globally asymptotically stable.

Lyapunov's direct method

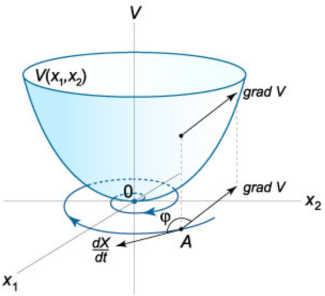


Figure : taken from: http://www.math24.net/

For physical systems: relate the physical energy with Lyapunov functions

Stability of passive systems

What about the stability of passive systems?

Stability of passive systems

Consider the system

$$\dot{x}(t) = f(x(t), u(t)), \qquad y(t) = h(x(t), u(t)), \tag{5}$$

with $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. Furthermore $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$. Let us addition define the **supply rate** $w(t) = w(u(t), y(t))$,

$$\int_0^t |w(u(\tau), y(\tau))d\tau| < \infty$$

Dissipative systems

The system (5) is said to be dissipative if there exists a so-called storage function $V(x) \geq 0$ such that the following dissipation inequality holds:

$$V(x(t)) \leq V(x(0)) + \int_0^t w(u(\tau), y(\tau))d\tau$$

along all possible trajectories of (5) starting at $x(0)$, for all $x(0)$, $t \geq 0$.

Passive systems

Passive systems

Suppose that the system (5) is dissipative with supply rate $w(u, y) = u^T y$ and storage function $V(x(t))$ with $V(0) = 0$; i.e. for all $t \geq 0$ we have that

$$V(x(t)) \leq V(x(0)) + \int_0^t u(\tau)^\top y(\tau) d\tau,$$

Then the system is passive.

Or, equivalently

$$\dot{V}(x(t)) \leq u^\top y$$

And in the absence of of an external input:

$\dot{V}(x(t)) \leq 0 \;\rightarrow$ **Passive systems are Lyapunov stable by definition**

Stabilization of passive systems

Questions:

- Can we make a passive system asymptotically stable?
- Can we increase the convergence rate to the origin?
- Can we shift the stable equilibrium?

Stabilization of passive systems

Let us consider system arising from some physical energy model. We then usually have

$$H(t) = H(t_0) + \underbrace{\int_0^t u(\tau)y(\tau)d\tau}_{\text{supplied energy}} - \underbrace{\int_0^t S(x(\tau))d\tau}_{\text{dissipated energy}} .$$

So if $H(x)$ qualifies as a Lyapunov function and $S(x)$ vanishes at $x = 0$ (and only in $x = 0$), then the system is asymptotically stable!

So why do we need the control then?

Stabilization of passive systems

- What if $S(x)$ vanishes for some $x \neq 0$ or $S(x) = 0$?: damping injection,
- What if we want to increase the rate of convergence?: damping injection,
- What if we want to stabilize at some different equilibrium point, $x = x^*$, $x^* \neq 0$: Energy shaping

Outline

1. Stability: definitions

2. Passivity based control: Damping injection and energy shaping

Stabilization of passive systems

- What if $S(x)$ vanishes for some $x \neq 0$ or $S(x) = 0$?: damping injection,
- What if we want to increase the rate of convergence?: damping injection,
- What if we want to stabilize at some different equilibrium point, $x = x^*$, $x^* \neq 0$: Energy shaping

Stabilization of passive systems: Damping injection

Consider the energy balance equation of a passive system:

$$H(t) = H(t_0) + \underbrace{\int_0^t u(\tau)y(\tau)d\tau}_{\text{supplied energy}} - \underbrace{\int_0^t S(x(\tau))d\tau}_{\text{dissipated energy}} .$$

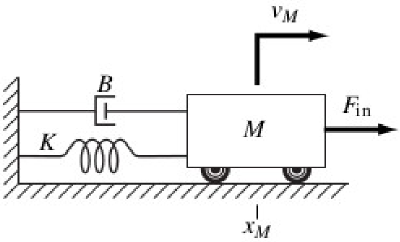
And assume that $H(x)$ qualifies as a Lyapunov function candidate. If we select the input $u = -Ky$, with K a positive definite constant matrix, then the energy balance equation becomes:

$$H(t) = H(t_0) \underbrace{-K\int_0^t y^2(\tau)d\tau}_{\text{controller}} - \underbrace{\int_0^t S(x(\tau))d\tau}_{\text{dissipated energy}} ,$$

$$H(t) = H(t_0) - \underbrace{\int_0^t \Big(Ky^2(\tau)d\tau + S(x(\tau))\Big)\, d\tau}_{\text{dissipated energy}}.$$

Example: mass-spring-damper system

Let us consider a simple linear translational MSD system:



Constitutive relations

$$F_s = F_{in}$$
$$F_B = Bv_B$$
$$p = Mv_M$$
$$q = K^{-1}F_K$$

Dynamic relations

$$F_M = \frac{dp}{dt}, \quad \text{or in integral form} \quad p(t) = p(t_0) + \int_0^t F_M(\tau)d\tau$$

$$v_K = \frac{dq}{dt}, \quad \text{or in integral form} \quad q(t) = q(t_0) + \int_0^t v_K(\tau)d\tau$$

Examples: the MSD system

$$H(x(t)) = \frac{1}{2}\frac{q}{K^{-1}}^2 + \frac{1}{2}\frac{p}{M}^2 \geq 0, \qquad H(0) = 0.$$

Hence H qualifies as a storage function and as a candidate Lyapunov function. Now,

$$H(t) = H(t_0) + \int_0^t F_{in}(\tau) v_M(\tau) d\tau - \int_0^t B v_M(\tau)^2 d\tau.$$

The system is passive if we choose $u = F_{in}$ and $y = v_M$, and furthermore, if we select $u = -Ky$, $(F_{in} = -Kv_M)$, then

$$\begin{aligned} H(t) &= H(t_0) - \int_0^t \left(K v_M^2(\tau) + B v_M^2(\tau) \right) d\tau. \\ &= H(t_0) - \int_0^t \underbrace{(K + B)}_{B'} v_M^2(\tau) d\tau \end{aligned}$$

We have changed (increased) the system's natural damping.

Stabilization of passive systems: Energy Shaping

Consider the energy balance equation of a passive system:

$$\underbrace{H(t) - H(t_0)}_{\text{stored energy}} = \underbrace{\int_0^t u(\tau)y(\tau)d\tau}_{\text{supplied energy}} - \underbrace{\int_0^t S(x(\tau))d\tau}_{\text{dissipated energy}} .$$

Assume that we want to change the closed-loop equilibrium to some forced (controlled) equilibrium $x = x^*$. In that case $H(x^*) \neq 0$, hence $H(x)$ can no longer be used as Lyapunov function!

We need to consider a new Lyapunov function candidate

Stabilization of passive systems: Energy shaping

Let us consider the energy balance equation and assume we have no dissipation

$$H(t) - H(t_0) = \underbrace{\int_0^t u(\tau)y(\tau)d\tau}_{\text{controller}}$$

The idea is to construct a new (closed-loop) energy function, by using the (state) feedback $u = \beta(x)$

$$H_d(x, x^*) = H(x) - \int_0^t \beta(x(\tau))y(\tau)d\tau$$

such that H_d, with $H_d(x^*) = 0$ qualifies as a Lyapunov function for the system.

Stabilization of passive systems: Energy shaping

If this function exist (yes! why should it exist?) it will be a state function such that

$$H_a(x, x^*) = -\int_0^t \beta(x(\tau))y(\tau)d\tau$$

Hence, $H_d(x) = H(x) + H_a(x)$. The time derivative of $H_d(x)$ along the trajectories of the system is given by

$$\dot{H}_d = \dot{H} + \dot{H}_a = \dot{H} + \frac{\partial H_a}{\partial x}^\top \dot{x}$$

$$\Rightarrow \frac{\partial H_a}{\partial x}^\top \dot{x} = -\beta(x)y$$

Hence, for dynamical systems of the form $\dot{x} = f(x, u)$, $y = h(x)$, in order to the function H_a to exist, the following PDE should be satisfied

$$\frac{\partial H_a}{\partial x}^\top (f(x, \beta(x))) = -\beta(x)h(x)$$

Stabilization of passive systems: Energy shaping

Some remarks

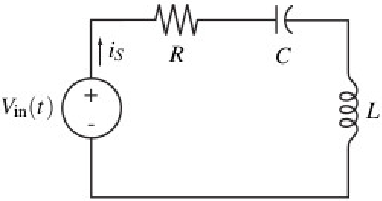
- Energy shaping requires the solution of a PDE: the matching equation. Not an easy task for general non-linear systems

$$H_a(x, x^*) = -\int_0^t \beta(x(\tau))y(\tau)d\tau$$

- The existence of solutions for the PDE is strongly related with the existence of physical invariants. In the case of port-Hamiltonian systems: Casimir functions.
- For systems arising from physical applications the energy shaping technique has been proven to be a powerful stabilization method.

Example: RLC circuit

Let us consider a simple linear RLC circuit:



Constitutive relations

$$u_s = V_{in}$$
$$u_r = RI_r$$
$$\phi = LI_L$$
$$Q = Cu_C$$

Dynamic relations

$$u_L = \frac{d\phi}{dt}, \quad \text{or in integral form} \quad \phi(t) = \phi(t_0) + \int_0^t u_L(\tau)d\tau$$

$$I_C = \frac{dQ}{dt}, \quad \text{or in integral form} \quad Q(t) = Q(t_0) + \int_0^t I_C(\tau)d\tau$$

Example: RLC circuit

The **state space model**

$$\frac{dQ}{dt} = \frac{\phi}{L}$$
$$\frac{d\phi}{dt} = -\frac{Q}{C} - R\frac{\phi}{L} + V_{in}$$

with state variables $x = [Q, \phi]$, output $y = \frac{\phi}{L} = \frac{x_2}{L}$ and input V_{in}. The energy of the system is given by

$$H(x) = \frac{1}{2}\frac{x_1}{C}^2 + \frac{1}{2}\frac{x_2}{L}^2$$

- If $V_{in} = 0$, the natural equilibrium is $x = (0, 0)$. If on other hand $V_{in} = V^*$, the forced equilibrium point is $x = (x_1^*, 0)$, with $x_1^* = CV^*$.

Example: RLC circuit

The matching equation becomes

$$\frac{\partial H_a}{\partial x}^{\top}(f(x,\beta(x))) = -\beta(x)h(x)$$
$$\frac{\partial H_a}{\partial x_1}\frac{x_2}{L} - \frac{\partial H_a}{\partial x_2}\left(\frac{x_1}{C} - R\frac{x_2}{L} - \beta(x)\right) = -\frac{x_2}{L}\beta(x)$$

Notice that the forced equilibrium corresponding to the x_2 coordinate already is a minimum of the physical energy $H(x)$, hence we only need to shape the closed-loop energy in the x_1 coordinate. Hence

$$\boxed{H_a = H_a(x_1)}$$

and the matching equation becomes

$$\frac{\partial H_a}{\partial x_1}\frac{x_2}{L} = -\frac{x_2}{L}\beta(x)$$

Hence, the function H_a exists if the feedback is chosen as $\boxed{\beta(x) = -\frac{\partial H_a}{\partial x_1}}$

Example: RLC circuit

Beautiful!

The matching equation (PDE) is automatically solved for any $H_a = H_a(x_1)$ provided that the state feedback is of the form $\beta(x) = -\frac{\partial H_a}{\partial x_1}$.

- It only remains to select $H_a(x_1)$ such that $H_d = H + H_a$ has a minimum at $x^* = (x_1^*, 0)$.

Recall that the open-loop energy function is

$$H(x) = \frac{1}{2}\frac{x_1}{C}^2 + \frac{1}{2}\frac{x_2}{L}^2$$

Hence if we chose

$$H_a(x_1) = \frac{1}{2C_a}x_1^2 - \left(\frac{1}{C_a} + \frac{1}{C}\right)x_1 x_1^*$$

The closed-loop energy function $H_d = H + H_a$

$$H_d(x) = \frac{1}{2}\frac{(x_1 - x_1^*)}{(C + C_a)}^2 + \frac{1}{2}\frac{x_2}{L}^2$$

Example: RLC circuit

$H_d(x, x^*)$ has a minimum at $x^* = (x_1^*, 0)$ if and only if $C_a > -C$

The resulting controller is

$$u = \beta(x) = -\frac{1}{2C_a} x_1^2 - \left(\frac{1}{C_a} + \frac{1}{C}\right) x_1 x_1^*$$

Final remarks

- We have revised some concepts from passivity based control techniques: Damping injection and Energy Shaping
- We have exploited the natural passivity of the system to design stabilizing controllers
- Works well in many applications, but.... we did not see the dissipation obstacle...

What remains for the last lesson

- Control by interconnection
- IDA-PBC
- Its application to PHS