Technical Report: An equivalence between data loss and SNR constraints

Sebastián A. Pulgar, Eduardo I. Silva*

February 10, 2012

Abstract

This paper focuses on two networked feedback system setups: (i) Multiple-input multiple-output (MIMO) linear time-invariant (LTI) systems with feedback over a set of MIMO i.i.d. erasure channels, and (ii) MIMO LTI systems with feedback over signal-to-noise ratio (SNR) constrained additive white noise channels. We show that, under suitable assumptions, both situations are equivalent from a second-order statistics point of view. This result establishes a fundamental relationship between the considered classes of networked systems, allowing one to address design problems involving either SNR constraints, or data dropouts, in a unified fashion. We illustrate our results by considering a static state feedback control problem.

1 Introduction

sec:intro

Networked control theory has received much attention in the recent literature [1]. This is due to its practical implications and also because of its theoretical challenges [1,13]. A key issue within the networked control paradigm is the understanding of the interplay between control objectives and communications constraints. This question has been addressed from several perspectives. For example, control problems subject to data-rate constraints have been explored in [15], the effects of data loss have been studied in [8,20,12,18], and SNR constraints have been treated in [3,19,17].

In this paper, we draw connections between two types of communications constraints, namely data dropouts and SNR constraints. Data loss is a consequence of transmission errors, fading, congestion or buffer overflows, which are prevalent in wireless communication links [13,11]. On the other hand, SNR constraints appear when dealing with analog communication channels [3,19]. Our goal is showing that analysis and design problems involving such constraints are, under suitable assumptions, equivalent from a second-order statistics point of view.

The idea of analyzing systems interconnected over unreliable channels by considering a statistically equivalent SNR constrained setup, can be traced back to [23]. In that work, the author studies state estimation problems for linear systems observed over an erasure channel. For such a setup, [23] constructs an auxiliary linear system that, when subjected to an instantaneous SNR constraint, yields signals with the same instantaneous second-order moments as in the original situation. Later, in [14], related results were presented for a given LTI feedback architecture involving one single-input single-output (SISO) erasure channel. In contrast to [23], the authors of [14] use a stationary approach and do not study instantaneous moments. The recent paper [20] extends [14] to arbitrary networked architectures involving one SISO erasure channel. The main contribution of [20] is an instantaneous and stationary second-order moment equivalence between feedback systems closed over SISO channels subject to either SNR constraints, or data dropouts.

Another connection between data loss and SNR constraints can be found in [9]. That paper presents a method for the design of predictive quantization schemes [10]. By approximating quantization errors

*The authors are with the Departamento de Electrónica, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile. Email: `eduardo.silva@usm.cl`

by an additive white noise source whose variance is subject to an SNR constraint, [9] notes that optimal designs hinge on the modified Riccati equation studied in [22] in the context of estimation subject to data loss. This observation, although not further explored in [9], reveals additional connections between estimation problems subject to either data dropouts or SNR constrains.

In this paper, we consider feedback architectures closed over either multiple MIMO i.i.d. erasure channels, or multiple MIMO SNR constrained channels. We first show that the instantaneous second-order moments of any signal in the former setup, are equal to the instantaneous moments of the corresponding signal in an auxiliary situation that arises when the erasure channels are replaced by gains, followed by SNR constrained additive white noise channels. As a second contribution, we show that the internal stability (in the usual sense [24]) of the auxiliary system subject to SNR constraints, and the satisfaction of a set of inequalities constraints, is equivalent to the mean square stability of the feedback system closed over erasure channels. Under a mild assumption, we show that the latter inequality constraints can be replaced by stationary SNR ones. Finally, we consider optimal design problems with a stationary quadratic cost. We show that optimal control problems over erasure channel are, essentially, equivalent to optimal control problems over SNR constrained additive noise ones.

The results presented in this paper unveil a fundamental relationship between networked control systems closed over either erasure or SNR constrained channels. They expand on the preliminary observations made in [23, 9], and extend [16, 20, 14] to cases where multiple MIMO channel are present.

To illustrate our findings, we study static state feedback control problems over erasure channels. We show that the SNR constrained perspective yields the same optimal synthesis equations as the approach presented in [2, Chapter 9], for the control of systems subject to multiplicative noise (see also [5]). Further illustrations of our results can be found in [20, 21]. A preliminary version of the results in this paper was presented in [16] for the two-channel case.

The remainder of this paper is organized as follows: Section 2 presents notation. Section 3 describes the considered setups, while Sections 4–6 present our main results. In particular, Section 4 presents the results related to instantaneous second-order moments, Section 5 presents results related to stability, and Section 6 studies optimal design problems. Section 7 presents an example, and Section 8 draws conclusions.

2 Notation

sec:notation

$\mathbb{R}$ stands for the reals and $\mathbb{N}_0$ for the non-negative integers. $\mathcal{P}\{*\}$ stands for the probability of $(*)$ and $\mathcal{E}\{*\}$ denotes the expectation of $(*)$. Given a matrix W, W^T denotes its transpose and W^H its conjugate transpose. If W is a square matrix, then $\rho(W)$ refers to its spectral radius. I_n denotes the $n \times n$ identity matrix, and $0_{n\times m}$ refers to an $n \times m$ zero matrix. The notation $\operatorname{diag}\{x_1, \ldots, x_n\}$, or $\operatorname{diag}\{x_i\}$, refers to a block diagonal matrix with diagonal blocks given by x_i.

We assume that all random processes are real valued and defined for $k \in \mathbb{N}_0$. We write x as shorthand for $\{x(k); k \in \mathbb{N}_0\}$. For any process x we define: $\mu_x(k) \triangleq \mathcal{E}\{x(k)\}$, $P_x(k) \triangleq \mathcal{E}\{(x(k) - \mu_x(k))(x(k) -\mu_x(k))^T\}$, $R_x(k+\tau, k) \triangleq \mathcal{E}\left\{(x(k+\tau) - \mu_x(k+\tau))(x(k) - \mu_x(k))^T\right\}$. We refer to $P_x(k)$ as the covariance matrix of x, and to $R_x(k+\tau, k)$ as the covariance function of x. We also define the stationary covariance matrix of x (when it exists) as $P_x \triangleq \lim_{k\to\infty} P_x(k)$. A random variable (process) is a second-order one if and only if it has finite mean and finite second-order moments (for all time instants $k \in \mathbb{N}_0$ and also when $k \to \infty$). We write $x \perp\!\!\!\perp y$ (resp. $x \perp y$) if and only if x and y are independent (resp. uncorrelated).

3 Setup and Assumptions

sec:setup

This paper aims at establishing a second-order moment equivalence between data dropouts and SNR constraints in networked control architectures. We first describe the setup involving data dropouts. To that end, we introduce the following definition (see also, e.g., [8, 20, 12, 18]):

:drop-channel

Definition 1 *A MIMO erasure channel is a device with input v_θ and output w_θ such that, for every $k \in \mathbb{N}_0$,*

$$w_\theta(k) = (\theta(k) + P)\, v_\theta(k), \tag{1}$$

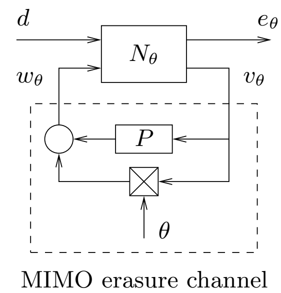

Figure 1: LTI system with feedback over a MIMO erasure channel. fig:N-mjls

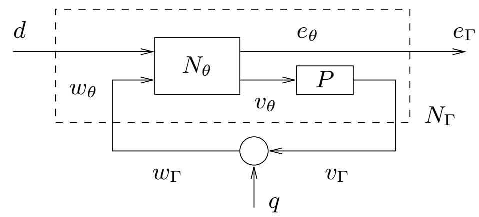

Figure 2: Auxiliary system that arises when, in Figure 1, one replaces the MIMO erasure channel by a gain P and a MIMO additive noise channel. fig:N-snr

where $\theta(k) \triangleq \operatorname{diag}\{\theta_i(k)I_{n_i}\}$, $P \triangleq \operatorname{diag}\{p_i I_{n_i}\}$, $i \in \{1,\dots,c\}$, n_i *is a positive integer,* θ_i *is a sequence of i.i.d. random variables,* $\theta_i(k) \in \{-p_i, 1-p_i\}$, $\mathcal{P}\{\theta_i(k) = 1-p_i\} = p_i \in (0,1)$, *and* $\theta_i \perp\!\!\!\perp \theta_j$ *for* $i \neq j$. $\square$

Erasure channels are idealized abstractions that allow one to capture the essential features of data loss [13,11]. We note that a more natural definition of erasure channel would be such that $w_\theta(k) = \tilde{\theta}(k)v_\theta(k)$, where $\tilde{\theta}(k) = \operatorname{diag}\left\{\tilde{\theta}_i(k)I_{n_i}\right\}$, and $\tilde{\theta}_i$ is as θ_i, except for the fact that it takes values in $\{0,1\}$ instead of $\{-p_i, 1-p_i\}$. We adopt the (equivalent) description in Definition 1 to streamline our derivations.

Consider the feedback loop of Figure 1, where N_θ is an LTI system described by

$$\begin{bmatrix} x_\theta(k+1) \\ e_\theta(k) \\ v_\theta(k) \end{bmatrix} = \begin{bmatrix} A & B_d & B_w \\ C_e & D_{de} & D_{we} \\ C_v & D_{dv} & 0 \end{bmatrix} \begin{bmatrix} x_\theta(k) \\ d(k) \\ w_\theta(k) \end{bmatrix}, \quad x_\theta(0) = x_o, \quad k \in \mathbb{N}_0, \tag{2}$$ eq:ve-Nt

x_θ is the corresponding n_x-dimensional state, x_o is the initial state, d models disturbances, e_θ is an n_e-dimensional output, and the link between v_θ and w_θ is given by a MIMO erasure channel. In (2), all signals are allowed to have arbitrary dimensions, and the real matrices (A, B_*, C_*, D_{**}) are of appropriate dimensions. Consistent with Definition 1, we introduce the partitions

$$v_\theta \triangleq \begin{bmatrix} v_{\theta_1}^T & \dots & v_{\theta_c}^T \end{bmatrix}^T, \quad w_\theta \triangleq \begin{bmatrix} w_{\theta_1}^T & \dots & w_{\theta_c}^T \end{bmatrix}^T, \quad B_w \triangleq \begin{bmatrix} B_{w_1} & \dots & B_{w_c} \end{bmatrix}, \tag{3}$$ eq:def-vwi

where v_{θ_i} and w_{θ_i} take values in $\mathbb{R}^{n_i}$, and $B_{w_i} \in \mathbb{R}^{n_x \times n_i}$.

We now describe an alternative setup involving additive noise channels.

ef:snr-channel **Definition 2** *A MIMO additive noise channel is a device with input* v_Γ *and output* w_Γ *such that, for every* $k \in \mathbb{N}_0$,

$$w_\Gamma(k) = q(k) + v_\Gamma(k), \tag{4}$$ eq:qp-1

where $q(k) \triangleq [\, q_1(k)^T \dots q_c(k)^T \,]^T$, q_i *is a zero-mean white noise sequence taking values in* $\mathbb{R}^{n_i}$, *and* $q_i \perp q_j$ *for* $i \neq j$. $\square$

The above additive noise channel is free of constraints and, as such, of no much interest by itself. In the remainder of the paper we will impose suitable constraints on the channel noise q to derive our results.

Given N_θ as in (2) and a MIMO erasure channel, we define the LTI system[1] $N_\Gamma \triangleq \operatorname{diag}\{I_{n_e}, P\} N_\theta$ described by

$$\begin{bmatrix} x_\Gamma(k+1) \\ e_\Gamma(k) \\ v_\Gamma(k) \end{bmatrix} = \begin{bmatrix} A & B_d & B_w \\ C_e & D_{de} & D_{we} \\ PC_v & PD_{dv} & 0 \end{bmatrix} \begin{bmatrix} x_\Gamma(k) \\ d(k) \\ w_\Gamma(k) \end{bmatrix}, \quad x_\Gamma(0) = x_o, \quad k \in \mathbb{N}_0, \tag{5}$$

eq:ve-Nsnr

where $(\cdot)_\Gamma$ has the same dimension as $(\cdot)_\theta$ and plays, for N_Γ, the role that $(\cdot)_\theta$ played for N_θ, P is as in Definition 1, and (A, B_*, C_*, D_{**}), d and x_o are as in (2). Consistent with (3), we also introduce the partitions

$$v_\Gamma \triangleq \begin{bmatrix} v_{\Gamma_1}^T & \dots & v_{\Gamma_c}^T \end{bmatrix}^T, \quad w_\Gamma \triangleq \begin{bmatrix} w_{\Gamma_1}^T & \dots & w_{\Gamma_c}^T \end{bmatrix}^T, \tag{6}$$

eq:def-gamma

where v_{Γ_i} and w_{Γ_i} take values in $\mathbb{R}^{n_i}$.

Figure 2 shows the feedback system that arises when one considers N_Γ and uses a MIMO additive noise channel as the link between v_Γ and w_Γ. We note that the same feedback scheme arises if one replaces, in Figure 1, the MIMO erasure channel by a gain P followed by a MIMO additive noise channel.

We will work under the following standard assumptions:

assu:xo-d

Assumption 1 *The initial state x_o is a zero-mean second-order random variable having covariance $P_o \geq 0$, d is a zero-mean second-order white noise sequence having covariance $P_d \geq 0$, $x_o \perp d$, $\theta \perp\!\!\!\perp (x_o, d)$, and $q \perp (x_o, d)$.* $\square$

4 Instantaneous Second-Order Moment Equivalence

sec:eqv-inst

In this section, we show that the first and second-order moments of the signals in the switched system of Figure 1 can be calculated by resorting to the analysis of the simpler LTI system of Figure 2.

teo:Rxs

Theorem 1 *Consider the switched system of Figure 1, where N_θ has the state space description in (2) and the link between v_θ and w_θ is given by a MIMO erasure channel. Also consider the LTI system of Figure 2, where N_Γ has the description in (5) with P as in Definition 1, and the link between v_Γ and w_Γ is given by a MIMO additive noise channel. If Assumption 1 holds, then $\mu_{x_\Gamma}(k) = \mu_{x_\theta}(k) = 0$ for every $k \in \mathbb{N}_0$ and if, in addition,*

$$P_{q_i}(k) = p_i^{-1}(1-p_i) P_{v_{\Gamma_i}}(k), \quad \forall k \in \mathbb{N}_0, \quad \forall i \in \{1, \dots, c\}, \tag{7}$$

eq:snr-e

then $R_{x_\Gamma}(k+\tau, k) = R_{x_\theta}(k+\tau, k)$ for every $k, \tau \in \mathbb{N}_0$.

Proof: The fact that $\mu_{x_\Gamma}(k) = \mu_{x_\theta}(k) = 0$ is immediate from (2), (5), and Assumption 1. To prove our second claim, one can exploit Assumption 1 and proceed as in the proof of Lemma 6.3 in [8] to show that the covariance matrix of the state of N_θ in Figure 1 satisfies

$$P_{x_\theta}(k+1) = A_o P_{x_\theta}(k) A_o^T + B_o P_d B_o^T + B_w \operatorname{diag}\left\{ p_i(1-p_i)\eta_i^T \left(C_v P_{x_\theta}(k) C_v^T + D_{dv} P_d D_{dv}^T \right) \eta_i \right\} B_w^T, \tag{8}$$

eq:teq2-4

where

$$A_o \triangleq A + B_w P C_v, \quad B_o \triangleq B_d + B_w P D_{dv}, \quad \eta_i^T \triangleq \begin{bmatrix} 0_{n_i \times (n_1 + \dots + n_{i-1})} & I_{n_i} & 0_{n_i \times (n_{i+1} + \dots + n_c)} \end{bmatrix}. \tag{9}$$

eq:def-eta

Consider now the auxiliary situation of Figure 2. Given (5), (4), Assumption 1, and the properties of q, we have that

$$P_{x_\Gamma}(k+1) = A_o P_{x_\Gamma}(k) A_o^T + B_o P_d B_o^T + B_w P_q(k) B_w^T. \tag{10}$$

eq:teq2-71

[1] We will abuse notation and also use N_i, $i \in \{\theta, \Gamma\}$, as an operator relating (d, w_i) with (e_i, v_i).

Also, we have that the covariance matrix of each component v_{Γ_i} of v_Γ in Figure 2 satisfies (recall (6))

$$P_{v_{\Gamma_i}}(k) = p_i^2 \eta_i^T \left(C_v P_{x_\Gamma}(k) C_v^T + D_{dv} P_d D_{dv}^T \right) \eta_i. \tag{11}$$

eq:pvs-lti

Thus, if (7) holds, then it follows from (8), (10) and (11) that both the state covariance matrix in the switched system of Figure 1, and the state covariance matrix in the LTI system of Figure 2, satisfy the same recursive equations. Since, in addition, $x_\theta(0) = x_\Gamma(0)$, then $P_{x_\Gamma}(k) = P_{x_\theta}(k)$ for every $k \in \mathbb{N}_0$.

To complete the proof, we note that it is straightforward to show that the state covariance functions in the switched system of Figure 1, and in the LTI system of Figure 2, are given by $R_{x_\theta}(k+\tau, k) = A_o^\tau P_{x_\theta}(k)$ and $R_{x_\Gamma}(k+\tau, k) = A_o^\tau P_{x_\Gamma}(k)$, respectively. Since $P_{x_\Gamma}(k) = P_{x_\theta}(k)$, our last claim follows. $\square$

The following immediate consequence of Theorem 1 relates the first and second-order moments of the output signals e_θ and e_Γ in the systems of Figures 1 and 2:

coro:Rys

Corollary 1 *Consider the setup and assumptions of Theorem 1. Then,* $\mu_{e_\Gamma}(k) = \mu_{e_\theta}(k) = 0$ *for every* $k \in \mathbb{N}_0$ *and if, in addition,* (7) *holds, then* $R_{e_\Gamma}(k+\tau, k) = R_{e_\theta}(k+\tau, k)$ *for every* $k, \tau \in \mathbb{N}_0$. $\square$

Theorem 1 and Corollary 1 state that the first and second-order moments of the state, and of any output of the switched system of Figure 1, can be studied by considering an auxiliary situation where the MIMO erasure channel has been replaced by a gain equal to P, followed by a MIMO additive noise channel subject to the instantaneous SNR constraints in (7) (see Figure 2). These results extend Theorem 1 and Corollary 1 in [16], and Lemma 9 in [20], to the multiple MIMO channel case.

5 Stability

sec:stability

The results presented above involve instantaneous moments only, providing no convergence or stability conditions. In this section, we show that there exists a relationship between the internal stability of the LTI system of Figure 2 (defined as usual [24]) and the following standard stability notion for the switched system of Figure 1 (see also [4]):

def:mss

Definition 3 *Consider the switched system of Figure 1, where* N_θ *is described by* (2) *and the link between* v_θ *and* w_θ *is given by a MIMO erasure channel. The resulting system is mean square stable (MSS) if and only if, for any initial state* x_o *and disturbance* d *satisfying Assumption 1, there exist* $\mu_{x_\theta} \in \mathbb{R}^{n_x}$ *and* $P_{x_\theta} \in \mathbb{R}^{n_x \times n_x}$, $P_{x_\theta} \geq 0$, *both not depending on* $(x_o, \theta(0))$, *such that* $\lim_{k\to\infty} \mu_{x_\theta}(k) = \mu_{x_\theta}$ *and* $\lim_{k\to\infty} P_{x_\theta}(k) = P_{x_\theta}$. $\square$

Our results make use of the following technical lemma:

lema:fact

Lemma 1 *Consider* θ *and* P *as in Definition 1 and matrices* $A, B_w, C_v, B_{w_1}, \ldots, B_{w_c}$ *as in* (2) *and* (3). *Define* $\delta_i \triangleq \mathcal{P}\{\theta(k) = T_i\}$ *and* $\mathcal{A}_i \triangleq A + B_w(T_i + P)C_v$, *where* $\{T_1, \ldots, T_{2^c}\}$ *is the set of all possible values for* $\theta(k)$. *Then, for any* $M \geq 0$ *of appropriate dimensions,*

$$\sum_{i=1}^{2^c} \delta_i \mathcal{A}_i M \mathcal{A}_i^T = (A + B_w P C_v) M (A + B_w P C_v)^T + \sum_{i=1}^{c} p_i(1-p_i) B_{w_i} \eta_i^T C_v M C_v^T \eta_i B_{w_i}^T. \tag{12}$$

eq:fact1

Proof: Consider the switched system

$$\tilde{x}(k+1) = (A + B_w(\theta(k) + P)C_v)\, \tilde{x}(k), \tag{13}$$

eq:mjls-aux

where $\tilde{x}(0)$ is second-order and $P_{\tilde{x}(0)} = M$. Under our assumptions, a standard manipulation [4, p. 32] shows that

$$P_{\tilde{x}}(k+1) = \sum_{i=1}^{2^c} \delta_i \mathcal{A}_i P_{\tilde{x}}(k) \mathcal{A}_i^T. \tag{14}$$

eq:varianza-li

On the other hand, a procedure similar to that employed to derive (8) allows one to conclude from (13) that

$$P_{\tilde{x}}(k+1) = A_o P_{\tilde{x}}(k) A_o^T + \sum_{i=1}^{c} p_i(1-p_i) B_{w_i} \eta_i^T C_v P_{\tilde{x}}(k) C_v^T \eta_i B_{w_i}^T, \tag{15}$$ eq:fact4

where A_o is as in (9) and we used (3). The result follows upon making $k = 0$ in both (14) and (15). □

We are now in a position to prove the main result of this section:

teo:est **Theorem 2** *Consider the setup and assumptions of Theorem 1. Then, the switched system of Figure 1 is MSS if and only if the LTI system of Figure 2 is internally stable and there exist* $W = \operatorname{diag}\{W_i\} > 0$*,* $W_i \in \mathbb{R}^{n_i \times n_i}$*, and* $Q \geq 0$ *such that, for every* $i \in \{1, \ldots, c\}$*,*

$$p_i(1-p_i)\eta_i^T C_v Q C_v^T \eta_i < W_i, \tag{16}$$ eq:teost-11

and

$$(A + B_w P C_v) Q (A + B_w P C_v)^T - Q + B_w W B_w^T = 0. \tag{17}$$ eq:teost-1

Proof:

1. Recall the definition of A_o and η_i in (9). If the LTI system is internally stable (i.e., if $\rho(A_o) < 1$), then there exists $Y > 0$ such that

   $$A_o Y A_o^T < Y. \tag{18}$$ eq:teost-2

   Also, given (16), there exists a sufficiently small $\epsilon > 0$ such that

   $$p_i(1-p_i)\eta_i^T C_v (Q + \epsilon Y) C_v^T \eta_i < W_i, \tag{19}$$ eq:teost-3

   where Q satisfies (17).

   Using the notation introduced in Lemma 1, we have that the switched system is MSS if and only if there exists an $M > 0$ such that (see Corollary 3.26 in [4])

   $$\sum_{i=1}^{2^c} \delta_i \mathcal{A}_i M \mathcal{A}_i^T < M. \tag{20}$$ eq:suma

   To complete this part of the proof we next show that (20) holds for $M = Q + \epsilon Y > 0$. Indeed, Lemma 1 yields

   $$\begin{aligned} \sum_{i=1}^{2^c} \delta_i \mathcal{A}_i (Q + \epsilon Y) \mathcal{A}_i^T &= A_o (Q + \epsilon Y) A_o^T + \sum_{i=1}^{c} p_i(1-p_i) B_{w_i} \eta_i^T C_v (Q + \epsilon Y) C_v^T \eta_i B_{w_i}^T \\ &\overset{(a)}{<} A_o (Q + \epsilon Y) A_o^T + \sum_{i=1}^{c} B_{w_i} W_i B_{w_i}^T \\ &\overset{(b)}{=} \epsilon A_o Y A_o^T + Q \\ &\overset{(c)}{<} Q + \epsilon Y, \end{aligned} \tag{21}$$ eq:desM

   where (a) follows from (19), (b) follows from (17) and the partition for B_w in (3), and (c) follows from (18).

2. If the switched system is MSS, then (20) and (12) imply that $\rho(A_o) < 1$ (recall the definitions in (9)) and, hence, the LTI system is internally stable. On the other hand, if the switched system is

MSS, then Corollary 3.26 and Theorem 3.33 in [4] imply that (again, we use the notation in Lemma 1)

$$X = \sum_{i=1}^{2^c} \delta_i \mathcal{A}_i X \mathcal{A}_i^T + V \tag{22}$$

eq:lyapunov-V

admits a unique solution X for any $V \geq 0$. Set $V = B_w \operatorname{diag}\left\{p_i(1-p_i)\eta_i^T \eta_i\right\} B_w^T \geq 0$. Given Lemma 1, (22) implies that

$$X = A_o X A_o^T + B_w \operatorname{diag}\left\{p_i(1-p_i)\eta_i^T C_v X C_v^T \eta_i\right\} B_w^T + B_w \operatorname{diag}\left\{p_i(1-p_i)\eta_i^T \eta_i\right\} B_w^T \tag{23}$$

eq:lyapuniov-X

admits a unique solution X. Set $W = \operatorname{diag}\{W_i\}$ with

$$W_i = p_i(1-p_i)\eta_i^T \left(C_v X C_v^T + I\right)\eta_i > p_i(1-p_i)\eta_i^T C_v X C_v^T \eta_i \geq 0 \tag{24}$$

eq:wi-candidat

(clearly, $W > 0$). We conclude from (23) and (24) that $W > 0$ satisfies (16) and (17) with $Q = X$. The proof is now complete. □

Remark 1 *Lemma 1 is key for proving Theorem 2 and the proof of that lemma exploits the instantaneous equivalence of Theorem 1. Thus, in contrast to Theorem 10 in [20], Theorem 2 is not an extension of the purely stationary approach used in [14]. If one were to use those ideas here, then one should be able to prove Lemma 1 using a direct algebraic approach, which seems rather difficult.* □

Theorem 2 shows that, under Assumption 1, the internal stability of the auxiliary LTI system of Figure 2, and the satisfaction of a set of inequality constraints, is equivalent to the MSS of the switched system of Figure 1. This result extends Theorem 10 in [20] to the multiple MIMO channel case.

We next present a result that serves as an alternative to Theorem 2. The result makes use of an additional assumption, but simplifies matters (see also Theorem 2 in [16] for a preliminary two-channel version).

d-alternativa **Corollary 2** *Consider the setup and assumptions of Theorem 1. Then:*

- *If the switched system of Figure 1 is MSS, then the system of Figure 2 is internally stable and the variance of q can be chosen to be a constant satisfying*[2]

$$P_{q_i}(k) = P_{q_i} = p_i^{-1}(1-p_i)P_{v_{\Gamma_i}}, \quad \forall i \in \{1, \ldots, c\}, \tag{25}$$

eq:pq-est

where $P_{v_{\Gamma_i}}$ denotes the stationary covariance of v_{Γ_i}.

- *If the LTI system of Figure 2 is internally stable, the variance of q satisfies* (25)*, and*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} T_{dv} P_d T_{dv}^H d\omega > 0, \tag{26}$$

eq:desigualdac

where $T_{dv}(e^{j\omega}) \triangleq C_v(e^{j\omega}I - A - B_w P C_v)^{-1}(B_d + B_w P D_{dv}) + D_{dv}$, then the switched system of Figure 1 is MSS.

Proof:

1. If the switched system of Figure 1 is MSS, then the LTI system of Figure 2 is internally stable (see Theorem 2). To complete the proof, we next show the existence of a choice for P_{q_i} satisfying (25). Set

$$P_{q_i} = P_{q_i}^{\theta} \triangleq p_i(1-p_i)\eta_i^T \left(C_v P_{x_\theta} C_v^T + D_{dv} P_d D_{dv}^T\right)\eta_i \geq 0\,, \tag{27}$$

eq:pqi-mjls

[2]Consistent with the notation introduced in Section 2, we omit here the argument k to refer to stationary covariance matrices.

where, since the switched system is MSS, P_{x_θ} is the unique solution of (see (8) and Corollary 3.26 in [4])

$$P_{x_\theta} = A_o P_{x_\theta} A_o^T + B_o P_d B_o^T + B_w \operatorname{diag}\left\{ p_i(1-p_i)\eta_i^T \left(C_v P_{x_\theta} C_v^T + D_{dv} P_d D_{dv}^T \right) \eta_i \right\} B_w^T . \tag{28}$$

eq:pxest-mjls

where A_o and B_o are as in (9). If P_{q_i} is as in (27), then, since the LTI system is internally stable, the state x_Γ of the LTI system of Figure 2 has a stationary covariance given by the unique solution to

$$P_{x_\Gamma} = A_o P_{x_\Gamma} A_o^T + B_o P_d B_o^T + B_w \operatorname{diag}\left\{ P_{q_i}^\theta \right\} B_w^T. \tag{29}$$

eq:pxest-lti

Given (27)–(29), and the unicity of both P_{x_θ} and P_{x_Γ}, we conclude that $P_{x_\Gamma} = P_{x_\theta}$. Thus, (27) becomes equivalent to $P_{q_i} = {p_i}^{-1}(1-p_i)P_{v_{\Gamma_i}}$ and the result follows.

2. If the LTI system of Figure 2 is internally stable and there exist P_{q_i} such that (25) holds, then

$$P_{q_i} = p_i^{-1}(1-p_i)P_{v_{\Gamma_i}} = p_i(1-p_i)\eta_i^T \left(C_v P_{x_\Gamma} C_v^T + D_{dv} P_d D_{dv}^T \right) \eta_i \tag{30}$$

eq:pqi-lti

where P_{x_Γ} satisfies (again, we use the definitions in (9))

$$P_{x_\Gamma} = A_o P_{x_\Gamma} A_o^T + B_o P_d B_o^T + B_w P_q B_w^T, \tag{31}$$

eq:recPx-st-mj

and $\rho(A_o) < 1$. Thus,

$$P_{x_\Gamma} = \sum_{i=0}^{\infty} A_o^i \begin{bmatrix} B_o & B_w \end{bmatrix} \operatorname{diag}\{P_d, P_q\} \begin{bmatrix} B_o & B_w \end{bmatrix}^T {A_o^i}^T . \tag{32}$$

eq:serie

Set $W_i = P_{q_i}$ and note that (30), (32) and (26) immediately yield

$$\begin{aligned} W_i &= p_i(1-p_i)\eta_i^T \left[C_v \left(\sum_{i=0}^{\infty} A_o^i B_o P_d B_o^T {A_o^i}^T \right) C_v^T + D_{dv} P_d D_{dv}^T + C_v \left(\sum_{i=0}^{\infty} A_o^i B_w P_q B_w^T {A_o^i}^T \right) C_v^T \right] \eta_i \\ &> p_i(1-p_i)\eta_i^T C_v \left(\sum_{i=0}^{\infty} A_o^i B_w P_q B_w^T {A_o^i}^T \right) C_v^T \eta_i \geq 0. \end{aligned} \tag{33}$$

We thus conclude that $W = \operatorname{diag}\{W_i\} = P_q > 0$ satisfies (16) and (17) with

$$Q = \sum_{i=0}^{\infty} A_o^i B_w P_q B_w^T {A_o^i}^T . \tag{34}$$

The proof is now complete. □

contraejemplo

Remark 2 *If, in Part 2 of Corollary 2, condition* (26) *is not satisfied, then it is not possible to ensure that the switched system of Figure 1 is MSS. Indeed, assume that* $c = 1$, $P_d = 0$, *and that* N_θ *in* (2) *is such that* $A = 0.7$, $B_w = 1$, $C_v = 0.55$ *(the remaining matrices are immaterial). If* $p = 0.5$, *then the LTI system of Figure 2 is internally stable and the unique* P_{q_1} *satisfying* (25) *is* $P_{q_1} = 0$. *However, since* $P_d = 0$, (26) *is not satisfied. For the above choice of parameters, the switched system of Figure 1 is not MSS, as the reader can readily verify (see also Remark 3.7 in [4]).* □

gualdad-clave

Remark 3 *In* (26), T_{dv} *is the closed loop transfer function between* d *and* v_Γ *in the architecture of Figure 2. Thus, satisfying condition* (26) *is equivalent to guaranteeing that the covariance matrix of the part of the signal* v_Γ *that is due to* d, *is positive definite. A sufficient condition for this to happen is that* $D_{dv} P_d D_{dv}^T > 0$. □

Corollary 2 states, under a rather mild assumption, that the switched system of Figure 1 will be MSS if and only if the LTI feedback system of Figure 2 is internally stable and the stationary SNR constraints in (25) are satisfied. A consequence of this result is explored in Section 6 below.

6 Stationary second-order Moment Equivalence and Optimal Designs

ptimal-designs

So far, we have studied the relationship between the instantaneous second-order moments of the signals in the feedback systems of Figures 1 and 2, and have also established a connection between the stability of both setups. In this section, we draw connections between optimal control problems for the considered architectures when a stationary performance measure is used. We begin with the following observation:

coro:eq-st

Corollary 3 *Consider the setup and assumptions of Theorem 1 and assume, in addition, that the switched system of Figure 1 is MSS and that the LTI system of Figure 2 is internally stable. If, instead of satisfying* (7), $P_{q_i}(k)$ *is constant and satisfies* (25), *then* $P_{x_\Gamma} = P_{x_\theta}$ *and* $P_{e_\Gamma} = P_{e_\theta}$.

Proof: If the switched system is MSS, then P_{x_θ} exists and is the unique solution to (28). If, on the other hand, the LTI system is internal stable and (25) holds, then P_{x_Γ} satisfies (31) with P_q as in (30). Equations (28) and (30)-(31) have an identical structure and, since (28) admits a unique solution, the result follows. □

Corollary 3 presents a stationary second-order moment equivalence between the systems of Figures 1 and 2. As such, it corresponds to the stationary counterpart of Theorem 1. Note that Corollary 3 requires the SNR constraint in (7) to hold in steady state only, and not for every time instant. (The latter stronger condition is necessary for the *instantaneous equivalence* presented in Theorem 1 and Corollary 1.) This result extends Corollary 11 in [20], and Corollary 2 in [16], to the multiple MIMO erasure channel case.

In order to establish a connection between the optimal design of the architectures of Figures 1 and 2, we introduce the set Ω containing all N_θ with the state space description in (2), and having a prescribed structure. The motivation behind the definition of Ω lies in the fact that, in most cases of interest, only certain sub-systems of N_θ can be designed whilst the remaining ones are fixed. For example, in a one degree-of-freedom dynamic output feedback control problem, where measurements are sent over an erasure channel, N_θ could be chosen so that $v_\theta = e_\theta = d + GCw_\theta$, where G is the (fixed) plant model and C is the (to be designed) controller.

Consider the setup of Figure 1 where the link between v_θ and w_θ is given by a MIMO erasure channel and Assumption 1 holds. Define

$$J_\theta(N_\theta) \triangleq \operatorname{trace}\{P_{e_\theta}\}, \quad J_\theta^{\text{opt}} \triangleq \inf_{N_\theta \in \mathcal{S}_\theta} J_\theta(N_\theta), \tag{35}$$

eq:def-J-theta

where $\mathcal{S}_\theta$ contains all N_θ in Ω that render the resulting switched system MSS. Also, consider the setup of Figure 2 where the link between v_Γ and w_Γ is given by a MIMO additive noise channel, and Assumption 1 holds. Define

$$J_\Gamma(N_\Gamma, P_{q_1^c}) \triangleq \operatorname{trace}\{P_{e_\Gamma}\}, \quad J_\Gamma^{\text{opt}} \triangleq \inf_{(N_\Gamma, P_{q_1^c}) \in \mathcal{F}} J_\Gamma(N_\Gamma, P_{q_1^c}), \tag{36}$$

eq:def-J-gamma

where $P_{q_1^c}$ is shorthand for $P_{q_1}, \ldots, P_{q_c}$, and $\mathcal{F}$ contains all $N_\Gamma = \operatorname{diag}\{I_{n_e}, P\} N_\theta$, with N_θ in Ω, and all positive-semidefinite $P_{q_1}, \ldots, P_{q_c}$ such that: (a) the resulting LTI feedback system is internally stable, (b) (25) holds, and (c) there exist $Q \geq 0$ and $W = \operatorname{diag}\{W_i\} > 0$ satisfying (16) and (17). In (36), the decision variables include the LTI system N_Γ and the channel noise variance matrices $P_{q_1}, \ldots, P_{q_c}$. The latter are to be chosen so as to satisfy the SNR constraints in (25).

lencia-diseos

Corollary 4 *Consider the setup and assumptions of Theorem 1. Then, the optimization problems in* (35) *and* (36) *are equivalent, i.e:*

1. *The optimization problem in* (35) *is feasible if and only if the optimization problem in* (36) *is feasible.*
2. *The optimal values of the optimization problems in* (35) *and* (36) *are equal, i.e.,* $J_\theta^{\text{opt}} = J_\Gamma^{\text{opt}}$.
3. *For any* $\epsilon > 0$, *if* $N_\theta \in \mathcal{S}_\theta$ *is such that* $J_\theta(N_\theta) < J_\theta^{\text{opt}} + \epsilon$, *then there exist* $P_{q_1^c}$ *such that*

$$(\operatorname{diag}\{I_{n_e}, P\} N_\theta, P_{q_1^c}) \in \mathcal{F} \quad \textit{and} \quad J_\Gamma(\operatorname{diag}\{I_{n_e}, P\} N_\theta, P_{q_1^c}) < J_\Gamma^{\text{opt}} + \epsilon. \tag{37}$$

Similarly, if $\epsilon > 0$ *and* $(N_\Gamma, P_{q_1^c}) \in \mathcal{F}$ *is such that* $J_\Gamma(N_\Gamma, P_{q_1^c}) < J_\Gamma^{\text{opt}} + \epsilon$*, then*

$$\operatorname{diag}\left\{I_{n_e}, P^{-1}\right\} N_\Gamma \in \mathcal{S}_\theta \quad \text{and} \quad J_\theta(\operatorname{diag}\left\{I_{n_e}, P^{-1}\right\} N_\Gamma) < J_\theta^{\text{opt}} + \epsilon. \tag{38}$$

Proof: Immediate from Theorem 2, Corollaries 2 and 3, and the relationship between N_Γ and N_θ (see (2), (5) and Figure 2). □

Corollary 4 states that the optimal design of the switched system of Figure 1 is equivalent to the optimal design of the LTI system of Figure 2, subject to both the stationary SNR constraints in (25), and the additional constraints in (16) and (17). The next result uses Corollary 2 to remove this additional constraints:

diseos-simple

Corollary 5 *Consider the setup and assumptions of Theorem 1. Assume, in addition, that there exists* $\delta > 0$ *such that any* $N_\theta \in \mathcal{S}_\theta$ *achieving* $J_\theta(N_\theta) < J_\theta^{\text{opt}} + \delta$*, and any* $(N_\Gamma, P_{q_1^c}) \in \mathcal{F}$ *achieving* $J_\Gamma(N_\Gamma, P_{q_1^c}) < J_\Gamma^{\text{opt}} + \delta$*, are such that* (26) *holds. Then, the optimization problem in* (35) *is equivalent to the problem of finding*

$$\hat{J}_\Gamma^{\text{opt}} \triangleq \inf_{\substack{N_\Gamma \in \mathcal{S}_\Gamma \\ 0 \le P_{q_i} < \infty \\ P_{v_{\Gamma_i}} = p_i(1-p_i)^{-1} P_{q_i}}} J_\Gamma(N_\Gamma, P_{q_1^c}), \tag{39}$$

eq:J-gamma-hat

where $\mathcal{S}_\Gamma$ *contains all* $N_\Gamma = \operatorname{diag}\{I_{n_e}, P\} N_\theta$*, with* N_θ *in* Ω*, that render the LTI system of Figure 2 internally stable.*

Proof: Immediate from Corollaries 2, 3, and their proofs. □

Remark 4 *In Corollary 5, the equivalence between the problems in* (35) *and* (39) *is to be understood,* mutatis mutandis*, in the sense described in Corollary 4. (We note that, in the present case, the claim made in the third point of the enumeration in Corollary 4 holds only for* $\epsilon \in (0, \delta)$*.)* □

Remark 5 *If the additional assumption in Corollary 5 does not hold, and the problem in* (35) *is feasible (resp. the problem in* (39) *is unfeasible), then the problem in* (39) *is also feasible (resp. the problem in* (35) *is also unfeasible). This is a consequence of Part 1 in Corollary 2. On the other hand, if the additional assumption in Corollary 5 does not hold, and the problem in* (35) *is unfeasible (resp. the problem in* (39) *is feasible), then no conclusions regarding the problem in* (39) *(resp. the problem in* (35)*) can be drawn from our results (see Remark 2).* □

Remark 6 *If the additional assumption in Corollary 5 does not hold, then one can always perturb the problem setup by adding white noise to the output* v_θ *of* N_θ *in both Figures 1 and 2. This guarantees, for the perturbed setup, the satisfaction of* (26) *at the cost of suboptimality (see also Remark 3).* □

Corollary 5 is, in our view, the most useful result in this paper. It states that optimal control problems involving MIMO LTI systems and a MIMO erasure channel are, essentially, equivalent to optimal control problems involving MIMO LTI systems and a MIMO additive noise channel subject to stationary SNR constraints. The latter problem setup includes the channel noise covariance matrices as decision variables, and is thus slightly non-standard. A solution strategy has been reported in [19] for the single SISO channel case. The static state feedback case is touched in Section 7 below. We also note that similar SNR constrained problems arise when designing optimal quantization schemes (see, e.g., [9, 10, 6] and the references therein).

Our results unveil a fundamental relationship between networked control system design problems over either erasure, or SNR constrained channels. They expand on preliminary observations made in [23, 9], and extend [16, 20, 14] to general feedback architectures involving multiple MIMO erasure channels. In particular, Corollary 5 generalizes an idea first explored by us in [20, Theorem 15] for a particular control architecture involving SISO channels (see also [21] for an application to estimation problems).

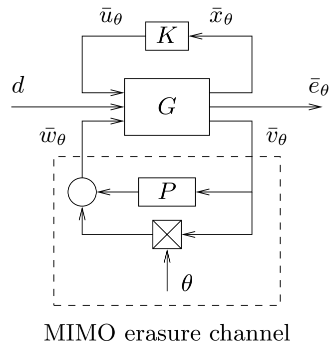

Figure 3: Networked control system comprising a MIMO erasure channel and a static state feedback control law. fig:general-MJ

7 An Application: Optimal State Feedback Designs

sec:state-fbck

This section considers a static state feedback optimal control problem, where communication takes place over a MIMO erasure channel. This problem can be solved by adapting results on the control of systems with multiplicative noise [2, 5]. We will recover such solution by using the SNR equivalence developed in previous sections.

Consider the networked control system of Figure 3, where G is described by[3]

$$\begin{bmatrix} \bar{x}_\theta(k+1) \\ \bar{e}_\theta(k) \\ \bar{v}_\theta(k) \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B}_d & \bar{B}_u & \bar{B}_w \\ \bar{C}_e & 0 & 0 & 0 \\ \bar{C}_v & 0 & \bar{D}_{uv} & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_\theta(k) \\ d(k) \\ \bar{u}_\theta(k) \\ \bar{w}_\theta(k) \end{bmatrix}, \quad \bar{x}_\theta(0) = x_o, \quad k \in \mathbb{N}_0, \tag{40}$$ eq:state-P-the

$\bar{x}_\theta$ is the $n_{\bar{x}}$-dimensional state, x_o is the initial state, d models disturbances, $\bar{u}_\theta$ is the $n_{\bar{u}}$-dimensional controller output, $\bar{e}_\theta$ is a signal related to closed loop performance, and the link between $\bar{v}_\theta$ and $\bar{w}_\theta$ is given by a MIMO erasure channel. In (40), all signals are allowed to have arbitrary dimensions, and $(\bar{A}, \bar{B}_*, \bar{C}_*, \bar{D}_{uv})$ are real matrices of appropriate dimensions. We consider a static state feedback control law, i.e., we assume that

$$\bar{u}_\theta(k) = K\bar{x}_\theta(k), \quad \forall k \in \mathbb{N}_0, \tag{41}$$ eq:ley-de-cont

where K is an $n_{\bar{u}} \times n_{\bar{x}}$ real matrix.

Our aim is to design the gain K so as to minimize the variance of $\bar{e}_\theta$ while guaranteeing the MSS of the resulting feedback system. Formally, we are interested in finding, for the setup described above and under Assumption 1,

$$\bar{J}_\theta^{\text{opt}} \triangleq \inf_{K \in \mathcal{K}_\theta} \operatorname{trace}\{P_{\bar{e}_\theta}\}, \tag{42}$$ eq:jopt-mjls

where $P_{\bar{e}_\theta}$ is the stationary covariance matrix of $\bar{e}_\theta$, and

$$\mathcal{K}_\theta \triangleq \{K \in \mathbb{R}^{n_{\bar{u}} \times n_{\bar{x}}} : \text{ the system of Figure 3 is MSS}\}. \tag{43}$$

It is fairly easy to proceed as in [2, Chapter 9] to show that the problem in (42) is feasible if and only if the following convex optimization problem is feasible: eq:lmis

$$\text{Find:} \quad \rho \triangleq \inf_{X,\Lambda,Z} \operatorname{trace}\{\Lambda\} \tag{44a}$$

[3]Given that our focus is on second-order statistics, we assume, without loss of generality, that there exists no direct feedthrough from d to $(\bar{e}_\theta, \bar{v}_\theta)$, nor from $(d, \bar{u}_\theta, \bar{w}_\theta)$ to $\bar{e}_\theta$.

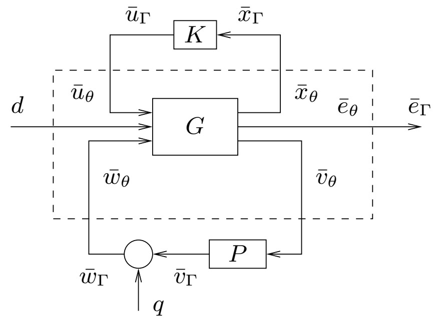

Figure 4: LTI system that arises when the MIMO erasure channel of Figure 3 is replaced by a gain P and a MIMO additive noise channel. fig:general-SN

$$\text{subject to:} \quad X > 0, \quad \Lambda > 0, \quad \begin{bmatrix} \Lambda & \bar{C}_e X \\ \star & X \end{bmatrix} > 0, \tag{44b}$$
eq:primeras-lm

$$\begin{bmatrix} X - \bar{B}_d P_d \bar{B}_d^T & \bar{A}_o X + \bar{B}_o Z & Y_1 & Y_2 & \cdots & Y_c \\ \star & X & 0 & 0 & \cdots & 0 \\ \star & \star & X & 0 & \cdots & 0 \\ \star & \star & \star & X & & \vdots \\ \star & \star & \star & \star & \ddots & 0 \\ \star & \star & \star & \star & \star & X \end{bmatrix} > 0, \tag{44c}$$
eq:segundas-lm

where $\bar{A}_o \triangleq \bar{A} + \bar{B}_w P \bar{C}_v$, $\bar{B}_o \triangleq \bar{B}_u + \bar{B}_w P \bar{D}_{uv}$, $Y_i \triangleq \sqrt{p_i(1-p_i)} \bar{B}_{w_i} \eta_i^T (\bar{C}_v X + \bar{D}_{uv} Z)$, $i \in \{1, \ldots, c\}$, $\star$ corresponds to entries that can be inferred by symmetry, η_i is as in (9), $\bar{B}_{w_i} \in \mathbb{R}^{n_{\bar{x}} \times n_i}$ are such that $\bar{B}_w = [\, \bar{B}_{w_1} \ldots \bar{B}_{w_c} \,]$, and the remaining matrices are as in (40). Moreover, if the problem in (42) is feasible and (X_o, Λ_o, Z_o) denotes the optimal value of (X, Λ, Z) in the optimization problem defined by (44), then $\bar{J}_\theta^{\text{opt}} = \rho$ and the optimal feedback gain K, say K_θ^{opt}, is such that $K_\theta^{\text{opt}} = Z_o X_o^{-1}$.

We next use Corollary 5 to show that one can arrive at the characterization of the solution to the problem in (42) presented in (44), by focusing on an SNR constrained optimal control problem. Consider the setup of Figure 4, where we have replaced the MIMO erasure channel in Figure 3 by a gain equal to P, followed by a MIMO additive noise channel (see also Figure 2). Define

$$\bar{J}_\Gamma^{\text{opt}} \triangleq \inf_{\substack{K \in \mathcal{K}_\Gamma \\ 0 \leq P_{q_i} < \infty \\ P_{\bar{v}_{\Gamma_i}} = p_i (1-p_i)^{-1} P_{q_i}}} \operatorname{trace}\{P_{\bar{e}_\Gamma}\}, \tag{45}$$
eq:prob-snr

where $i = 1, \ldots, c$, $P_{\bar{e}_\Gamma}$ is the stationary covariance matrix of $\bar{e}_\Gamma$, $\mathcal{K}_\Gamma \triangleq \{K \in \mathbb{R}^{n_{\bar{u}} \times n_{\bar{x}}}$: the LTI system of Figure 4 is internally stable, i.e., $\rho(\bar{A}_o + \bar{B}_o K) < 1\}$, and $p_1, \ldots, p_c$ are the successful transmission probabilities of the MIMO erasure channel.

For any feasible $(K, P_{q_1^c})$, we have that the stationary variances of $\bar{x}_\Gamma$, $\bar{e}_\Gamma$ and $\bar{v}_\Gamma$ in Figure 4 satisfy

$$P_{\bar{x}_\Gamma} = (\bar{A}_o + \bar{B}_o K) P_{\bar{x}_\Gamma} (\bar{A}_o + \bar{B}_o K)^T + \bar{B}_d P_d \bar{B}_d^T + \bar{B}_w P_q \bar{B}_w^T, \tag{46}$$
eq:var-x-ej1

$$P_{\bar{e}_\Gamma} = \bar{C}_e P_{\bar{x}_\Gamma} \bar{C}_e^T, \tag{47}$$
eq:var-e-ej1

$$P_{q_i} = p_i^{-1}(1-p_i) P_{\bar{v}_{\Gamma_i}} = p_i(1-p_i) \eta_i^T (\bar{C}_v + \bar{D}_{uv} K) P_{\bar{x}_\Gamma} (\bar{C}_v + \bar{D}_{uv} K)^T \eta_i. \tag{48}$$
eq:var-Pq-ej1

By making the SNR constraints in (48) explicit, (46) reduces to

$$\begin{aligned} P_{\bar{x}_\Gamma} &= (\bar{A}_o + \bar{B}_o K) P_{\bar{x}_\Gamma} (\bar{A}_o + \bar{B}_o K)^T \\ &\quad + \sum_{i=1}^{c} p_i (1-p_i) \bar{B}_{w_i} \eta_i^T (\bar{C}_v + \bar{D}_{uv} K) P_{\bar{x}_\Gamma} (\bar{C}_v + \bar{D}_{uv} K)^T \eta_i \bar{B}_{w_i}^T + \bar{B}_d P_d \bar{B}_d^T. \end{aligned} \tag{49}$$
eq:var-x-ej2

Assume now that, when restated in terms of the current setup and variables, the additional assumptions introduced in Corollary 5 hold. Then, Corollary 2 guarantees that any (sufficiently close to) optimal choice for $(K, P_{q_1^c})$ renders the switched system of Figure 3 MSS. We can thus focus, without loss of generality, on such $(K, P_{q_1^c})$. For any such choice of parameters, the operator Ψ, defined via

$$\Psi(M) \triangleq (\bar{A}_o + \bar{B}_o K) M (\bar{A}_o + \bar{B}_o K)^T + \sum_{i=1}^{c} p_i(1-p_i) \bar{B}_{w_i} \eta_i^T (\bar{C}_v + \bar{D}_{uv} K) M (\bar{C}_v + \bar{D}_{uv} K)^T \eta_i \bar{B}_{w_i}^T, \quad (50)$$

eq:operador

is stable (i.e., for any $M > 0$, $\Psi^k(M) \to 0$ as $k \to \infty$; see Corollary 3.26 and Theorem 3.33 in [4], and also Lemma 2.1 in [5]). Hence, (49) admits a unique solution for $P_{\bar{x}_\Gamma}$ given by $P_{\bar{x}_\Gamma} = \sum_{i=0}^{\infty} \Psi^i(\bar{B}_d P_d \bar{B}_d^T)$ [5].

The facts in the above paragraph allow one to use a standard manipulation (see, e.g., [7, Section 6.4]) to show, starting from (47) and (49), that the optimization problem in (45) is equivalent to the one of finding

:pre-lmis-snr

$$\inf_{X,\Lambda,Z} \operatorname{trace}\{\Lambda\} \quad (51a)$$

$$\text{subject to:} \quad X > 0, \quad \Lambda > 0, \quad \Lambda > \bar{C}_e X \bar{C}_e^T, \quad (51b)$$

eq:primeras-lm

$$X > (\bar{A}_o + \bar{B}_o K) X (\bar{A}_o + \bar{B}_o K)^T + \sum_{i=1}^{c} p_i(1-p_i) \bar{B}_{w_i} \eta_i^T (\bar{C}_v + \bar{D}_{uv} K) X (\bar{C}_v + \bar{D}_{uv} K)^T \eta_i \bar{B}_{w_i}^T + \bar{B}_d P_d \bar{B}_d^T. \quad (51c)$$

eq:segundas-lm

The above problem can be shown to be equivalent to the problem in (44) upon using Schur complements [7, 2], a trivial congruence transformation, and by defining $Z \triangleq KX$. We have thus verified Corollary 5 for the state feedback architecture considered here.

Example 1 *Assume that G in* (40) *is such that*

$$\bar{A} = \begin{bmatrix} 0.8 & 0 \\ 0 & 1.2 \end{bmatrix}, \quad \bar{B}_d = \bar{B}_w = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \bar{B}_u = 0, \quad \bar{C}_e = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad \bar{C}_v = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \bar{D}_{uv} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and that d is zero-mean white noise with unit covariance matrix. For this choice of parameters, we solved the problem in (42) *by using the formulation in* (44)*. The results are shown in Figure 5, where the optimal cost $\bar{J}_\theta^{\text{opt}}$ is shown as a function of (p_1, p_2). In Figure 5, we also plot an estimate of* $\operatorname{trace}\{P_{\bar{e}_\theta}\}$ *obtained by simulating the equivalent (stationarily) SNR constrained system. (The plots corresponds to averages over one hundred 10^4-samples-long simulations.) The simulation results match our theoretical findings quite well. We also compared the simulated stationary channel SNRs in each channel with the corresponding theoretical values, namely with $p_i(1-p_i)^{-1}$ (not shown here). Again, the simulation results matched our theoretical ones.* □

c:conclusions

8 Conclusions

This paper has established a second-order moment equivalence between networked control systems subject to either data loss or SNR constraints. We have explored both the instantaneous and stationary regimes, as well as stability properties, yielding a set of results that allow one to deal with many situations of interest. The relevance of our results is twofold. First, it simplifies the analysis and design of networked control systems subject to data loss, allowing one to use elementary LTI system analysis and design tool. Second, it allows one to immediately translate results valid in either of the considered networked setups, into results valid in the other (see also [20, 21]).

Further work should focus on more general networked design problems. Also relevant is the use of our results to draw further connections between the existent literature on control over unreliable channel (e.g., [8, 12, 18]) and that on control subject to SNR constraints (e.g., [3, 19, 17]).

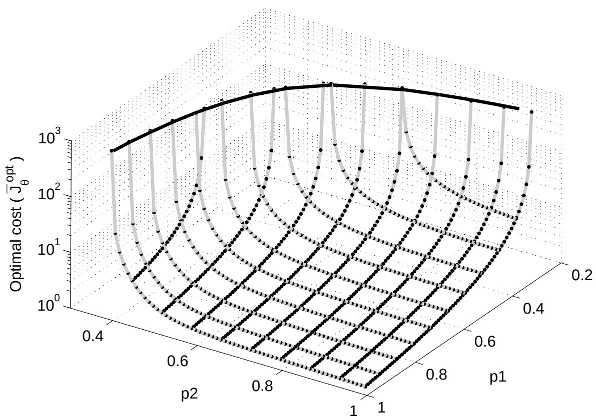

Figure 5: Optimal cost $\bar{J}_\theta^{\mathrm{opt}}$ as a function of (p_1, p_2) (black dots), and estimated cost obtained by simulating the equivalent SNR constrained setup (gray lines). The black auxiliary curve is such that, when projected into the (p_1, p_2) plane, separates the successful transmission probabilities that allow one achieve MSS, from those that do not. `fig:resultados`

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