Images, Geometric Hydrodynamics, and Shape Evolution

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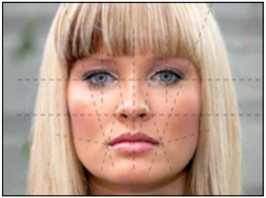
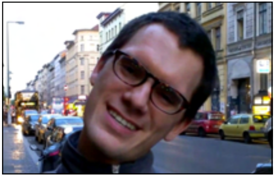
Outline

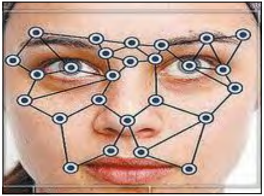
A problem inspired by Image Registration

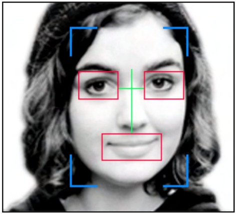
How similar/different are 2 images/shapes?

FACES





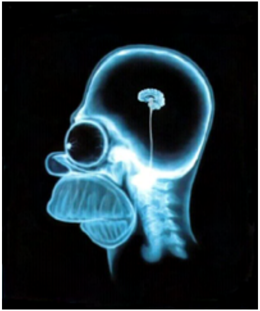


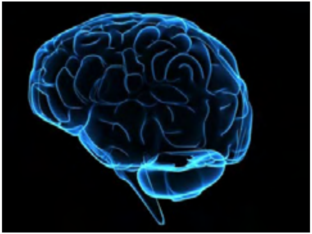


A problem inspired by Image Registration

How similar/different are 2 images/shapes?

Brain

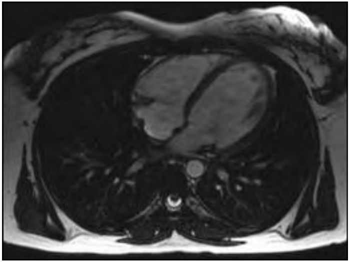


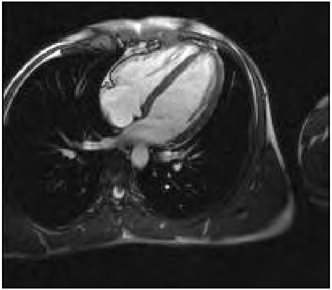


A problem inspired by Image Registration

How similar/different are 2 images/shapes?

HEART

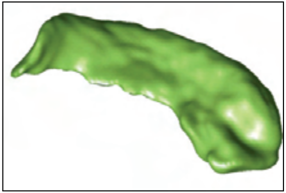




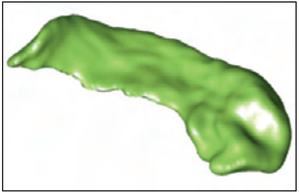
A problem inspired by Image Registration

How similar/different are 2 images/shapes?

Hypothalamus



Healthy



Schizophrenia

Assuming *a correlation between healthy state or diseased, and the shape of the anatomical structure, Can we define a distance between these shapes?*

A problem inspired by Image Registration:

Image Registration (Alignment): Estimation of **optimal transformation** between: Images, Points (Landmarks), Curves, Edges, Scalar functions, etc.

- **Intensity** $\begin{cases} \text{Cross-correlation,} \\ \text{Mutual Information,} \\ \text{Sum of Squared Intensity Differences,} \\ \text{Ratio image uniformity.} \end{cases}$

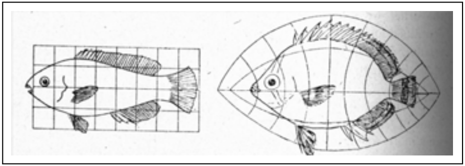
- **Models** $\begin{cases} \underline{\text{Linear}}\text{: Rigid, Affine, Piecewise Affine.} \\ \underline{\text{Non-Rigid}}\text{: Radial basis (thin plate, surface splines),} \\ \text{Viscous fluids, Elastic, Large deformations.} \end{cases}$

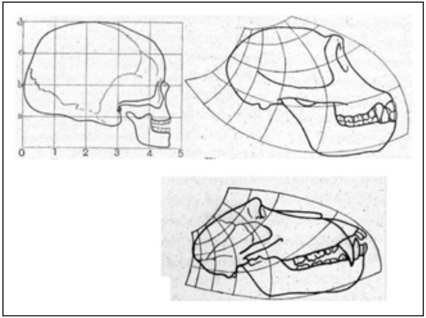
Outline

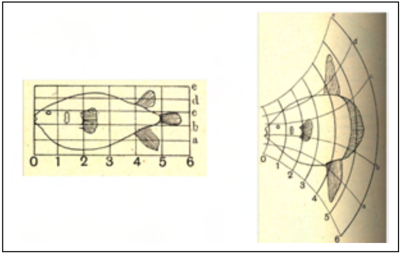
Building Geodesics: Background

D'Arcy Thompson: "On Growth and Form" (1917)

"To what extent is it true that individuals of closely related species can be transformed, one into the other, by a conformal transformation which carries every significant feature of one into the corresponding feature of the other?"







Building Geodesics: Background

- D'Arcy Thompson: "On Growth and Form" (1917).
- Ulf Grenander Mathematical rigor: Use of Lie group action.
- A. Trouve, *An infinite dimensional group approach for physics based models in patterns recognition*, International Journal of Computer Vision, 1995. (D. Ebin and J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math, 1970.)
- Diffeomorphic matching framework, M. Miller, A. Trouve, L. Younes, *On the metrics and Euler Lagrange equations of computational anatomy*. Annual Review of Biomedical Engineering, 2002.
- M. Bruveris, F. Gay-Balmaz, D.D. Holm, T.S. Ratiu, *The momentum map representation of images*, Journal of Nonlinear Science, 2011. (Geometric Mechanics)

Building Geodesics: Generalized Euler Equations (V. Arnold)

- Lie group G with Lie algebra $\mathfrak{g}$, acting on a general manifold $\mathcal{M}$.
- (Right-Left) Invariant Lagrangian on TG: $\mathcal{L} : TG \to \mathbb{R}$.
- Euler-Poincaré reduction theorem, construct a reduced Lagrangian $l : u \in \mathfrak{g} \to \mathbb{R}$

$$\overbrace{\underbrace{\delta \int_a^b \mathcal{L}(\varphi, \dot{\varphi})\, dt = 0}_{Ham.'s\ Principle}}^{E-L\ equations} \Leftrightarrow \overbrace{\underbrace{\delta \int_a^b l(u)\, dt = 0}_{Red.\ Ham.'s\ Principle}}^{E-P\ equations}$$

$$\begin{tikzcd}
\boxed{E-L \text{ on } TG} \arrow[rr, leftrightarrow, "Legendre\ Tr."] \arrow[d, leftrightarrow, "{\begin{matrix}E-P\\ Red./Reconst.\end{matrix}}"'] & & \boxed{Ham. \text{ on } T^*G} \arrow[d, leftrightarrow, "{\begin{matrix}L-P\\ Red./Reconst.\end{matrix}}"] \\
\boxed{E-P \text{ on } \mathfrak{g}} \arrow[rr, leftrightarrow, "Reduced\ Legendre\ Tr."] & & \boxed{L-P \text{ on } \mathfrak{g}^*}
\end{tikzcd}$$

- The kinetic $\mathcal{L}$ defines a metric on G, so $E-P$ represents a geodesic motion on G.

Generalized Euler Equations: $SO(3)$

Rotation of rigid bodies in $\mathbb{R}^3 \longrightarrow$ Geodesic motion in $SO(3)$ Lie group:

$L - P$ equation:

$$\boxed{SO(3):\ \frac{d\,\Pi}{dt} - ad^*_{\Omega}\Pi = 0} \qquad \text{where} \quad ad^*_{\Omega}\Pi = \Pi \times \Omega$$

Generalized Euler Equations: $\mathit{Diff}_{\mu}(\Omega)$

- Ideal fluid confined on $\Omega \subset \mathbb{R}^3$, configuration space $\mathit{Diff}_{\mu}(\Omega)$.
- Position of every particle $X \in \Omega$: $\varphi(t, X)$, $\varphi \in \mathit{Diff}_{\mu}(\Omega)$.
- Right-Invariant Lagrangian on $T\mathit{Diff}_{\mu}(\Omega)$: $\mathcal{L} : T\mathit{Diff}_{\mu}(\Omega) \to \mathbb{R}$.
- With this setup, Ebin and Marsden proved well-possessedness for incompressible Euler's equations, starting the field of geometric hydrodynamics.

Generalized Euler Equations: *Diff* (Ω) : *EPDiff equation*

$$\partial_t \boldsymbol{m} + \boldsymbol{u} \cdot \nabla \boldsymbol{m} + (\nabla \boldsymbol{u})^T \cdot \boldsymbol{m} + \boldsymbol{m}\,(div\,\boldsymbol{u}) = 0$$

or

$$\partial_t \boldsymbol{m} + ad^*_u \boldsymbol{m} = 0$$

$\boldsymbol{m}$: Momentum corresponding to the vector field u,
ad^*: Coadjoint action of the Lie algebra $\mathfrak{g}$ of $Diff(\Omega)$ on $\mathfrak{g}^*$.

- When $\Omega = S^1$, KdV equation: Geodesic flow with respect to the right invariant L^2 metric on the Bott-Virasoro group.
- When $\Omega = S^1$, Camassa-Holm (C-H) equation: Geodesic flow with respect to the right invariant H^1 metric on the Bott-Virasoro group. (Existence of geodesics has been proven for this PDE).
- In 1-D, *EPDiff* is:

$$m_t + u\,m_x + 2m\,u_x = 0 \qquad \text{with} \qquad m = (1 - \alpha^2 \partial_x^2)u, \tag{1}$$

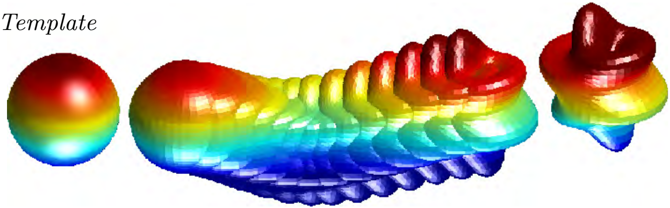
i.e., the dispersionless limit of the C-H equation for shallow water motion, with **train of peakons** solution.

Generalized Euler Equations: $\mathit{Diff}(\Omega)$: EPDiff *equation*

Generalized Euler Equations: $\mathit{Diff}(\Omega)$: EPDiff *equation*

Outline

Diffeomorphic Registration: Numerical Model



$$\boxed{\partial_t m + ad_v^* m = 0} \quad \Leftrightarrow \quad \boxed{\partial_t \begin{Bmatrix} \varphi(t,y) \\ \mu(t) \end{Bmatrix} = \mathcal{F}_{\varphi,\mu} \begin{Bmatrix} \varphi(y,t) \\ \mu(t) \end{Bmatrix}}$$

EPDiff (Geodesics on *Diff*) $\Leftrightarrow$ Hamiltonian System

Diffeomorphic Registration: Numerical Model

Energy on the initial momentum: $E(\rho_0) = (\,\rho_0 |\, K\rho_0) + \lambda U(\varphi(1))$
Variation on the initial momentum: $\rho_0 \to \rho_0 + \delta\rho_0$

$$\delta E(\rho_0) = 2\,(\,\delta\rho_0 |\, K\rho_0) + \lambda \left(\frac{\delta U}{\delta\varphi}(\varphi(1)) \middle|\, \delta\varphi(1) \right) \tag{1}$$

1) From *EPDiff* ... Linearized model:

$$\boxed{\partial_t \begin{Bmatrix} \delta\varphi \\ \delta\mu \end{Bmatrix} = \mathcal{J}_{\varphi,\mu} \begin{Bmatrix} \delta\varphi \\ \delta\mu \end{Bmatrix}} \qquad \delta\varphi(0) = 0, \delta\mu(0) = \delta\rho_0$$

2) Adjoint system: $\xi_\varphi := \delta\varphi^*$, $\xi_\mu := \delta\mu^*$

$$\boxed{\partial_t \begin{Bmatrix} \xi_\varphi \\ \xi_\mu \end{Bmatrix} = -\mathcal{J}^*_{\varphi,\mu} \begin{Bmatrix} \xi_\varphi \\ \xi_\mu \end{Bmatrix}} \qquad \xi_\varphi(1) = \frac{\delta U}{\delta\varphi}(\varphi(1)), \xi_\mu(1) = 0$$

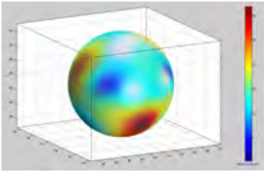
3) So (1) becomes

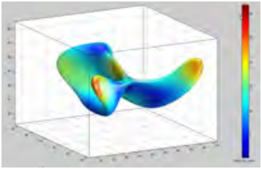
$$\delta E(\rho_0) = (\,\delta\rho_0 |\, 2K\rho_0 + \lambda\xi_\mu(0))$$

$$\Rightarrow \boxed{\nabla E(\rho_0) = 2\rho_0 + \lambda K^{-1}\xi_\mu(0)} \qquad \Rightarrow \boldsymbol{\rho_0}^{n+1} = \boldsymbol{\rho_0}^{n} - \varepsilon\,\nabla E(\rho_0^n)$$

Diffeomorphic Registration: Numerical Model

Template

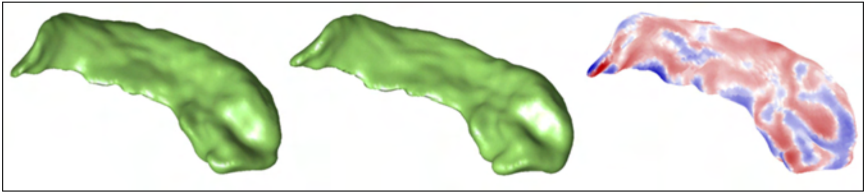


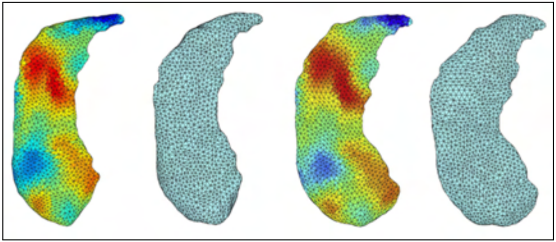


Target

Felipe Arrate. "*Evolution Equations On The Group Of Diffeomorphisms, With Applications In Computational Anatomy*".

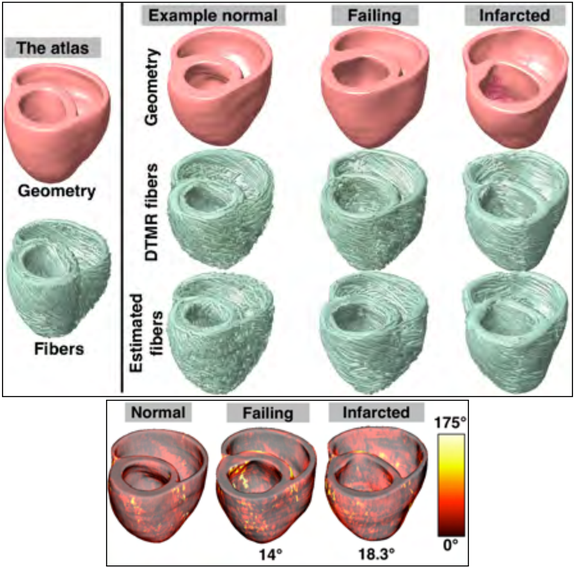
Examples: Medical Images Registration





L. Younes, Felipe Arrate, M.I. Miller: "*Evolution equations in computational anatomy*". NeuroImage 45 (2009).

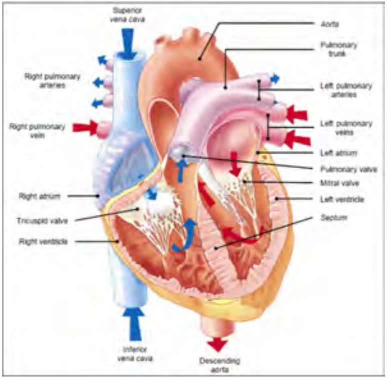
Examples: Miocardial Fibers



Vadakkumpadan, Felipe Arrate, T. Ratnanather, R. Winslow, May 2011.

Outline

Problem: Cardiovascular diseases



- Heart diseases are today responsible for the 28% of deaths in western countries (Cancer is 30%), around $400,000$ deaths per year in Europe
  - fatigue
  - insufficient ventricular contraction
- Sudden Death affects 1 in $10,000$ per year in developed countries
  - electrical disorder
  - ventricular fibrillation

GOAL

Classification of pathologic hearts by individualized model of cardiac motion and electrophysiology.

- Diagnosis.
- Efficacy of defibrillation in infarcted hearts.
- Ablation location for correcting arrhythmias.

Problem: How can we classify pathological heart states?

1) Understand the physics and background on the problem

- Heart Mechanics involves a detailed knowledge of elastic constants at different levels inside the myocardium.
- Importance of trabecular fibers.
- Electrophysiology.
- Detailed description of boundary conditions.

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2) "Brainstorming"

- Is Aortic pressure enough? Can we accurately (to a good level) get those measurements?
- Attempts using electrocardiogram measurements have their own line of research, but rough approximations.
- Mechanics?. Difficult measurements. Proposed models have more or less detail on the tensor structure of the fibers, stress tensor, ...
- Motion Learning? → Images (cine CT, MRI).
  http://www.youtube.com/watch?v=lZZvPgquif4
  http://www.youtube.com/watch?v=rhallml-juk

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3) Motion Learning

- What can we infer from a study mainly based on medical images?
- Is it possible to infer good/bad behaviour just from deformation?.
- Do we need to develop new Mathematics?.

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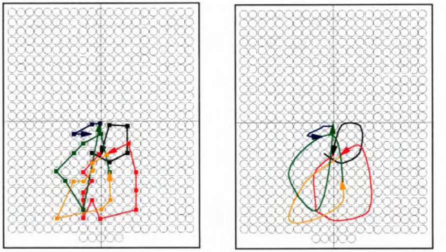
- Importance of trabecular fibers.
- Electrophysiology.
- Detailed description of boundary conditions.

3) Motion Learning

- What do we follow, Tip of the heart? "Patterns of spiral tip motion in cardiac tissues", Kim, D.T. and Kwan, et al., Chaos, 8, 1, 1998

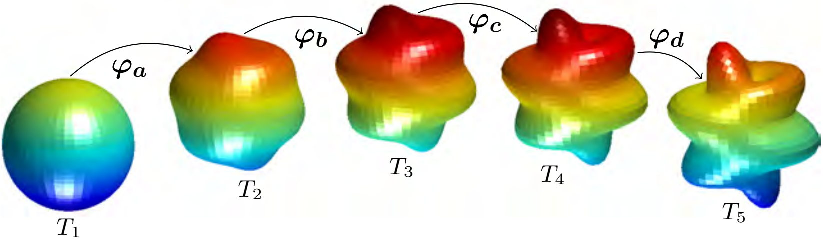


Problem: How can we classify pathological heart states?

3) Motion Learning: Start SIMPLE...

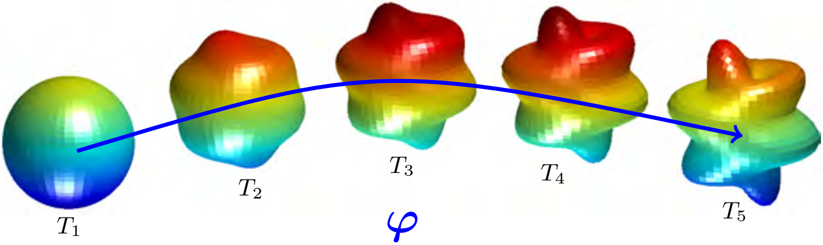
- 2D.
- Left ventricle.

Tracking Shape Deformation

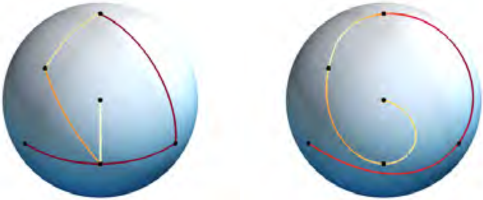


$$\begin{Bmatrix} y_1 \\ \rho_1 \end{Bmatrix} \qquad \begin{Bmatrix} y_2 \\ \rho_2 \end{Bmatrix} \qquad \begin{Bmatrix} y_3 \\ \rho_3 \end{Bmatrix} \qquad \begin{Bmatrix} y_4 \\ \rho_4 \end{Bmatrix} \qquad \begin{Bmatrix} y_5 \\ \rho_5 \end{Bmatrix}$$

Tracking Shape Deformation

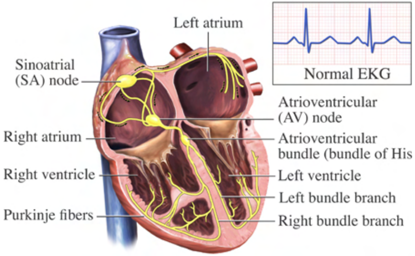


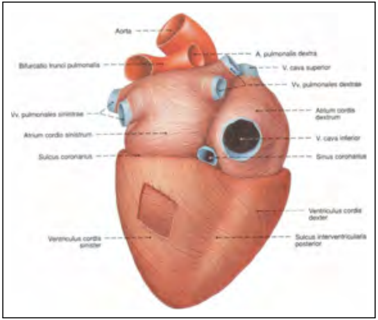
$$\begin{Bmatrix} y_1 \\ \rho_1 \\ \dot{\rho}_1 \end{Bmatrix}$$



Meier, D., D.D. Holm, F. Gay-Balmaz, F.X. Vialard, T. Ratiu, *Invariant higher-order variational problems*, Communications in Mathematical Physics, 2011.

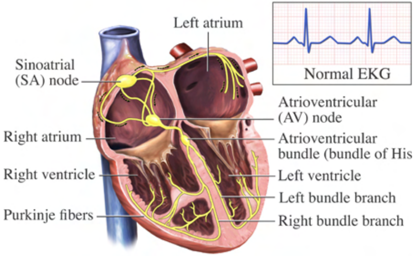
Outline

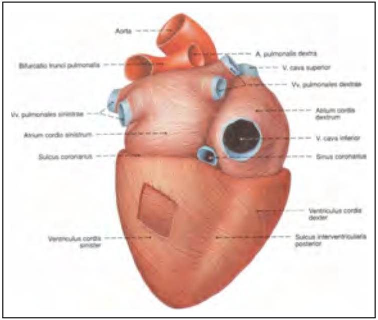




Maxwell's equations, but...

1. Electrical field strengths are not too high $\Rightarrow$ biological tissue is assumed to behave linear with regard to its electrical properties.
2. Cardiac electrical activity is reflected by low frequency components only ($\leq$ several kHz) $\Rightarrow$ derivatives with respect to time can be neglected ('quasi-static' approximation of Maxwell's equations)





Maxwell:

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$J = -DE$$

and...

$$\frac{\partial B}{\partial t} = 0 \Rightarrow E = -\nabla u$$

$\longrightarrow$

Monodomain Model:

1) $\dfrac{du}{dt} = c_1 f(u,w) + \nabla \cdot (D\nabla u)$

2) $\dfrac{dw}{dt} = \epsilon(u - \gamma w)$

3) $f(u,w) = c_1 u(u-\alpha)(u-1) + c_2 uw$

FitzHugh-Nagumo Model

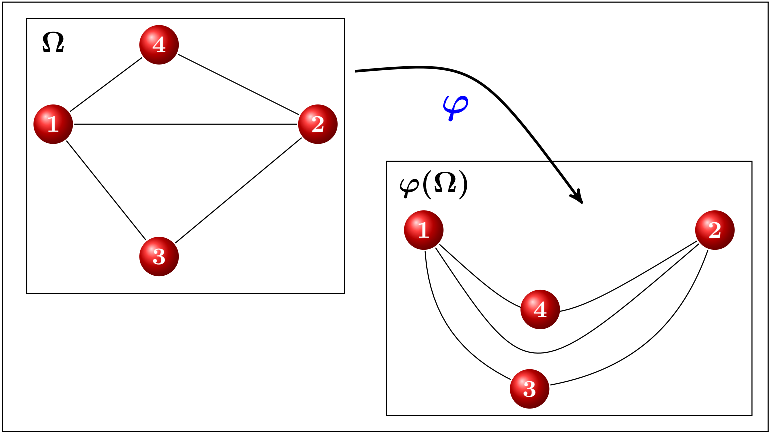
$$I_{ion} = f(u,w)$$

$$\frac{dw}{dt} = \epsilon(u - \gamma w)$$

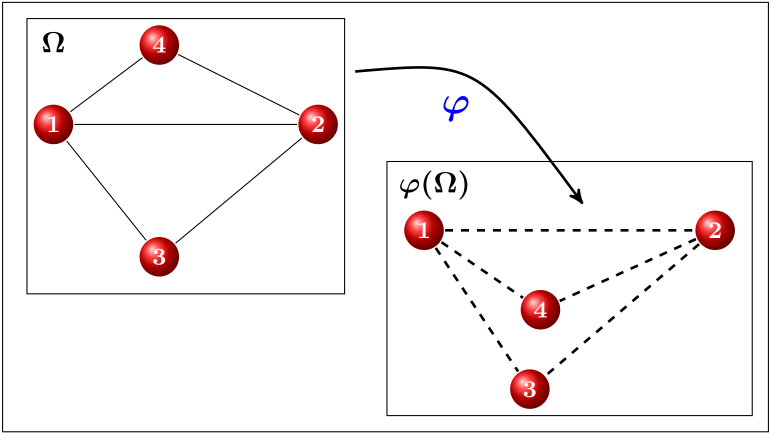
Mesh or ... Meshless Method?

- Finite element methods have been extensively used for the spatial discretization of the myocardium.
- Complicated meshing procedures and element-based interpolation functions often result in algorithms which are either easy to implement, but numerically inaccurate, or accurate but labor-intensive
- The meshfree platform is more adaptive to different cardiac geometries and thus beneficial to individualized analysis.

AND...



Ω
4
1
2
3
φ
$\varphi(\Omega)$
1
2
4
3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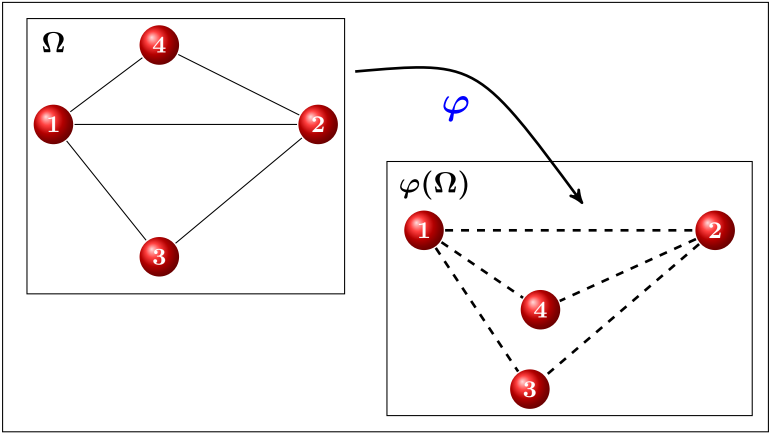

Particle Methods

- Complicated volume meshing procedures are excluded.
- No re-meshing is needed for improving spatial accuracy when deformation occurs.

Moving Least Squares (MLS) Approximation

1. $\{x_1(t), \ldots, x_N(t)\}$ nodes (particles) in $\Omega \subset \mathbb{R}^3$
2. $\mathbf{p}^T(x) = [p_1(x), \ldots, p_m(x)]$ polynomial basis.

Governing equation

Given:

- *locations* $x_i(t)$, *for* $i = 1, \ldots, N$,
- *values* $u(x_i, t)$, *for* $i = 1, \ldots, N$

Solve the associated ODE system

$$\frac{d}{dt}\begin{Bmatrix} u \\ w \end{Bmatrix} = \mathbf{\Phi}(u, w)$$

and obtain

- *new locations* $x_i(t + \Delta t)$, *for* $i = 1, \ldots, N$,
- *new values* $u(x_i, t + \Delta t)$, *for* $i = 1, \ldots, N$

Moving Least Squares

- *Approximate the solution by:*

$$u(x) = \sum_{k=1}^{m} p_k(x) a_k(x)$$

- *Minimizing the functional*

$$\mathcal{J} = \sum_{i=1}^{N} w(x - x_i) \left[\mathbf{p}^T(x_i)\mathbf{a}(x) - u_i\right]^2$$

($w(x - x_i)$ *weighing function with compact support)*

- *Solve the MLS problem*

$$A(x)\mathbf{a}(x) = B(x)\mathbf{u}$$

MLS Approximation: Monodomain Model

1. Monodomain model

u - membrane potential
w - recovery variable.

$$\frac{\partial u}{\partial t} = f(u,w) + \nabla \cdot (D \nabla u)$$
$$\frac{\partial w}{\partial t} = \epsilon(u - \gamma w)$$
$$f(u,w) = c_1 u(u-\alpha)(u-1) - c_2 uw$$

2. Weak formulation

ϕ - regular test function

$$\int_\Omega \phi \frac{\partial u}{\partial t} = \int_\Omega \phi\, f(u,w) - \int_\Omega \nabla \phi^T (D \nabla u)$$
$$\int_\Omega \phi \frac{\partial v}{\partial t} = \int_\Omega \phi\, \epsilon(u - \gamma w)$$

3. Meshfree approximation

$\Phi = [\phi_1(x), \ldots, \phi_N(x)]$ - *shape function*

$$u \sim \Phi \mathbf{u} \qquad w \sim \Phi \mathbf{w}$$
$$\left[\int_\Omega \Phi^T \Phi\right] \frac{\partial \mathbf{u}}{\partial t} = \left[\int_\Omega \Phi^T \Phi\right] f(\mathbf{u},\mathbf{w}) - \left[\int_\Omega \nabla \Phi^T D \nabla \Phi\right] \mathbf{u}$$
$$\left[\int_\Omega \Phi^T \Phi\right] \frac{\partial \mathbf{w}}{\partial t} = \left[\int_\Omega \Phi^T \Phi\right] \epsilon(\mathbf{u} - \gamma \mathbf{w})$$

4. ODE system

$$\frac{\partial \mathbf{u}}{\partial t} = f(\mathbf{u},\mathbf{w}) + M^{-1} K\, \mathbf{u}$$
$$\frac{\partial \mathbf{w}}{\partial t} = \epsilon(\mathbf{u} - \gamma \mathbf{w})$$
$$f(\mathbf{u},\mathbf{w}) = c_1 \mathbf{u} \circ (\mathbf{u} - \alpha) \circ (\mathbf{u} - 1) - c_2 \mathbf{u} \circ \mathbf{w}$$

where

$$M = \int_\Omega \Phi^T \Phi \qquad K = \int_\Omega \nabla \Phi^T D \nabla \Phi$$

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2. Weak formulation

ϕ - regular test function

$$\int_\Omega \phi \frac{\partial u}{\partial t} = \int_\Omega \phi\, f(u,w) - \int_\Omega \nabla \phi^T (D \nabla u)$$
$$\int_\Omega \phi \frac{\partial v}{\partial t} = \int_\Omega \phi\, \epsilon(u - \gamma w)$$

W R O N G! ! !

MLS Approximation: Monodomain Model

1. Monodomain model

u - *membrane potential*
w - *recovery variable.*

$$\frac{\partial u}{\partial t} = f(u,w) + \nabla \cdot (D\nabla u)$$
$$\frac{\partial w}{\partial t} = \epsilon(u - \gamma w)$$
$$f(u,w) = c_1 u(u-\alpha)(u-1) - c_2 uw$$

2. Weak formulation

ϕ - *regular test function*

$$\int_\Omega \phi \frac{\partial u}{\partial t} = \int_\Omega \phi\, f(u,w) - \int_\Omega \nabla\phi^T (D\nabla u)$$
$$\int_\Omega \phi \frac{\partial v}{\partial t} = \int_\Omega \phi\, \epsilon(u - \gamma w)$$

3. Meshfree Approximation

$$M\frac{\partial \mathbf{u}}{\partial t} + \left[\int_\Omega \Phi^T \left[(J\Phi)\dot{\mathbf{x}}\right]^T\right]\mathbf{u} = M f(\mathbf{u},\mathbf{w}) - K\mathbf{u}$$
$$M\frac{\partial \mathbf{w}}{\partial t} + \left[\int_\Omega \Phi^T \left[(J\Phi)\dot{\mathbf{x}}\right]^T\right]\mathbf{w} = M\epsilon(\mathbf{u} - \gamma\mathbf{w})$$

...Some preliminary results... fixed heart

Thanks for listening...