Analysis and design of partly networked architectures for two-input two-output LTI systems (Technical Report) *

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1. ABSTRACT

This article deal with the control of a two-input two-output (TITO) discrete time LTI plant, where one control signal is to be sent through a transparent channel, and the remaining control signal has to be sent through a channel that presents signal to noise ratio (SNR) constraints. For this situation, we study three different control architectures: state feedback, output feedback, and output feedback enriched with feedback around the constrained channel. We explore how the stability, and its best achievable performance, depend on the location of the SNR constrained channel for each control architectures under study. An interesting conclusion of this work is that the best channel choice, when only stability is to be met, may be inadequate when the focus is on performance. A numerical example is included to illustrate our results.

Keywords: Networked control systems, signal-to-noise ratio, optimal control, stabilization.

2. INTRODUCTION

Standard control system theory considers that communication links between plant and controller are transparent. However, this assumption is not always adequate since, in practice, these links may have communication constraints such as signal-to-noise ratio (SNR) constraints, data-rate constraints, or may be subject to data loss or random delays (see e.g., Nair et al. (2007); Schenato et al. (2007); Braslavsky et al. (2007); Krtolica et al. (1994)). Systems where these communication constraints take place are known as networked control systems (NCSs) (Antsaklis and Baillieul (2007)).

This paper deal with discrete time LTI control loops closed over SNR constrained channels. A key feature of such a setup is that the corresponding results allows one to gain insights that are valid in more realistic scenarios, including control over erasure channels or digital channels subject to average data-rate constraints (Silva et al. (2011); Silva and Pulgar (2011)).

Previous results on control over SNR constrained channels have established that unstable poles, non-minimum phase (NMP) zeros, and the relative degree of the plant, are the main factors that limit the minimal SNR compatible with mean square stabilization (Braslavsky et al. (2007); Rojas (2009)). It is also known that in the state feedback case, stability can be achieved with SNRs that are in general smaller than in the output feedback case (Braslavsky et al. (2007)). Other schemes include the case of output feedback control using a feedback channel (Silva et al. (2010)). In that case one recovers the minimal SNR compatible with stability that applies in the state feedback case, provided the feedback channel is ideal except for a one-step delay.

Once stability is guaranteed, the next natural step is to characterize the best achievable performance while satisfying a given channel SNR constraint. Freudenberg and Middleton (2008) and Freudenberg et al. (2008) addressed that problem, for minimum phase plants of relative degree one, using the plant output variance at a terminal instant as a performance measure. These works show that the best achievable performance can be achieved using a linear time-varying control scheme. However, the transient performance achievable with the scheme of Freudenberg and Middleton (2008) may be poor. An LTI strategy is considered by Silva et al. (2010) for output feedback architectures. In that work, standard 2-norm properties are used to formulate optimal control problems subject to SNR constraints as a two-stage optimization problem, involving a standard optimal control problem subject to a quadratic inequality constraint, and a coupled line search.

All works cited above deal with single-input single-output (SISO) plants. For multiple-input multiple-output (MIMO) plants the results are less extensive. In Li et al. (2009) analytic expressions are obtained for both stability requirements and optimal performance when the loop is closed over a power constrained MIMO channel. That work uses a two degree of freedom controller scheme to control a minimum phase plant when the total power of the channel input is constrained. A numerical approach based on LMIs is presented in Pulgar et al. (2011), where a general MIMO plants subject to multiple SNR constraints are controlled using a state feedback scheme.

This paper consider a TITO plant model that has to be controlled using two channels: one transparent channel and one SNR constrained channel. This kind of NCS is referred as partly networked control system since only some of its communication links are constrained (Silva et al. (2008)). For the setup described above, we are interested in gaining insight into how the scalar SNR constrained channel affects the control of the TITO plant. Considering this aim, we would like to determine which one of the control signals should be sent over the transparent channel so as to guarantee stability with the least SNR in the non-transparent channel, or so as to guarantee the best performance using a given SNR constrained channel. We study those problems for three control architectures: state feedback control, output feedback control, and output feedback with channel feedback. Interestingly, our results show that the best channel choice when only stability is to be met, may be inadequate when the focus is on performance.

The paper is organized as follows: Section 2 present the notation used and some preliminary results. Section 3 states the problems of interest in this paper. Stabilization problems are treated in Section 4, while Section 5 deal with performance issues. An example is shown in Section 6. Conclusion and future research are given in Section 7.

3. NOTATION AND PRELIMINARIES

$\mathbb{N}_0 \triangleq \{0, 1, ...\}$, $\mathbb{R}^+ \triangleq \{n \in \mathbb{R} : 0 < n < \infty\}$, $\phi \triangleq \{\}$. A diagonal matrix whose non zero elements are $a_1, a_2, ..., a_n$ is represented by $\text{diag}\{a_1, a_2, ..., a_n\}$. Given any matrix F, $F_{\ell *}$ (resp. $F_{*\ell}$) denotes its ℓ^{th} row (resp. column), its transpose is denoted by F^T and its pseudoinverse (right or left according to the context) by $F^\dagger$. Given any scalar f, $|f|$ denotes its magnitude and $\bar{f}$ its complex conjugate. The notation $\overline{(\cdot)}$ is also used to distinguish between related but different variables, however no confusion should arise.

Consider a random process $\{x(k) : k \in \mathbb{N}_0\}$. To simplify the notation we sometimes omit the dependence on k and write just x. If x is wide sense stationary (wss), then the stationary variance of x and its spectral factor are denoted by σ_x^2 and Ω_x, respectively. We say that x is second order process (or variable) if its first and second moments are finite for any $k \in \mathbb{N}_0$ and when $k \to \infty$.

Denote as $\mathcal{R}$ the set of all real rational discrete-time transfer functions. The following sets are subsets of $\mathcal{R}$: $\mathcal{R}_p$ contains all proper transfer function, $\mathcal{R}_{sp}$ contains all strictly proper transfer function, $\mathcal{RH}_\infty$ contains all stable and proper transfer functions, $\mathcal{RH}_2$ contains all stable and strictly proper transfer functions and $\mathcal{RH}_2^\perp$ contains all transfer function that have no poles inside or on the unit circle. Consider $H(z)$ a transfer function with no poles on the unit circle. Then $H(z)$ can be decomposed as $H(z) = [H(z)]_{\mathcal{H}_2} + [H(z)]_{\mathcal{H}_2^\perp}$, where $[H(z)]_{\mathcal{H}_2} \in \mathcal{RH}_2$ and $[H(z)]_{\mathcal{H}_2^\perp} \in \mathcal{RH}_2^\perp$. A zero c of $H(z)$ is a complex number such that $H(c)$ loses rank, and it is a finite NMP zero if $1 < |c| < \infty$. The relative degree of $H(z)$ is defined as the number of zeros at infinity, while the 2-norm is denoted by $\|H(z)\|_2$ (Zhou et al. (1996)). We use $H(z) = (A, B, C, D)$ as a shortcut to denote a standard state space representation of $H(z)$ given by matrices A, B, C and D. For simplicity, the dependence of z in $H(z)$ is sometimes omitted.

4. PROBLEM FORMULATION

Consider a TITO discrete time LTI plant having minimal state space description

$$x(k+1) = A_G x(k) + B_1(\tilde{u}_1(k) + d_1(k)) + B_2(\tilde{u}_2(k) + d_2(k)) \tag{1a}$$

$$y(k) = C_G x(k), \tag{1b}$$

where x is the n_x-dimensional state, y is the output, $\tilde{u}_1$ and $\tilde{u}_2$ are the two (scalar) control inputs, and $d \triangleq [d_1 \; d_2]^T$ is a disturbance. The plant is to be controlled by a proper LTI controller $K(z)$ using two communication channels between the controller and the plant. One of these channels is transparent, while the second one is an SNR constrained additive white noise (AWN) channel:

***Definition** 1.* An SNR constrained AWN channel is a device with scalar input v and scalar output w such that, $\forall v(k) \in \mathbb{R}, k \in \mathbb{N}_0$,

$$w(k) = v(k) + q(k), \tag{2}$$

where q is a zero mean white noise sequence whose variance σ_q^2 is a design variable[2] that is to be chosen subject to the stationary SNR constraint

$$\gamma \triangleq \frac{\sigma_v^2}{\sigma_q^2} \leq \Gamma, \tag{3}$$

where Γ corresponds to the maximum admissible SNR in the channel. ■

For the situation described above, we would like to determine which of the controller outputs should be sent over the transparent channel and which one through the SNR constrained AWN channel. Denote the controller output as $u \triangleq [u_1 \; u_2]^T$. Fix $\ell \in \{1, 2\}$. If u_ℓ is sent over the constrained channel, then $\tilde{u} \triangleq [\tilde{u}_1 \; \tilde{u}_2]^T = \tilde{q}_\ell + u$, where

$$\tilde{q}_\ell \triangleq \eta_\ell q, \tag{4}$$

with q is as in Definition 1 and η_ℓ corresponds to the ℓ^{th} column of the 2×2 identity matrix. For this choice, the AWN channel input v from (2) equals u_ℓ and the AWN channel output w equals $\tilde{u}_\ell$.

We will consider three control schemes, as described below:

- **Case 1: State Feedback.** In this case, we assume that the state x of the plant can be measured without noise, as depicted in Figure 1.a), where $\overline{G}(z) \triangleq (A_G, [B_1 \; B_2], I, 0)$ is the transfer function from the input $\tilde{u}$ to the state x, $G(z) \triangleq C_G \overline{G}(z)$ is the transfer function from $\tilde{u}$ to the output y, u is the controller output, and $\tilde{q}_\ell$ is as in (4).
- **Case 2: Output Feedback.** In this case, only the plant output y can be measured and we use the control scheme of Figure 1.b), where all the symbols are as in Case 1.
- **Case 3: Output Feedback with channel feedback.** We also consider a third case, where output feedback is combined with a delayed feedback path around the AWN channel. The considered setup is depicted in Figure 1.c), where all symbols are as before, and $\tilde{w} \triangleq z^{-h} \eta_\ell^T \tilde{u}$, with $h \in \mathbb{N}_0$ corresponding to the feedback channel delay.

[2] This is possible using pre and post scaling factors around power constrained additive noise channels (see Silva et al. (2010)).

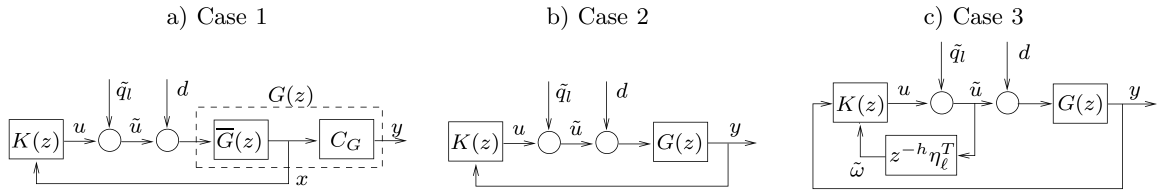

Fig. 1. Control arquitectures over a scalar SNR contrained AWN channel. a) Case 1: State Feedback Control Loop. b) Case 2: Output Feedback Control Loop. c) Case 3: Output Feedback Control Loop with Feedback Channel

Remark 1. In general, the existence of a transparent feedback channel does not seem plausible since that would imply that it is preferable to send v through this channel rather than over the constrained one. However, in some practical setups, the existence of channel feedback does not imply the existence of a physical channel. Indeed, when erasure channels are considered, the presence of packet acknowledgements is equivalent to having the channel output w available at the controller side (Schenato et al. (2007)). ■

We are now ready to formally state the first problem to be addressed in this paper:

Problem 1. Fix $\ell \in \{1,2\}$, consider the setups of Case i, $i \in \{1,2,3\}$, and assume that:

- The state space description of $G(z)$ in (1) is such that $(A, [B_1\ B_2])$ is stabilizable and (C, A) is detectable.
- $G(z)$ has finite NMP zeros and unstable poles of algebraic multiplicity one.
- $G(z)$ has no poles or zeros on the unit circle.
- The initial state of the plant is a second order random variable uncorrelated with (d, q).
- The disturbance d is a wss second order process uncorrelated with q and having spectral factor Ω_d.

Find

$$\gamma^i_{\inf} \triangleq \inf_{\substack{K\in S_i \\ \sigma_q^2 \in \mathbb{R}^+}} \gamma, \tag{5}$$

where S_i denotes the set of all proper LTI controllers that stabilizes the loop considered in Case i. ■

For LTI systems, and assuming second order initial states and second order wss inputs, internal stability is equivalent to mean square stability (MSS; Åström (1970)). Thus, solving Problem 1 amounts to finding the minimal SNR compatible with MSS.

It follows from the definition of $\gamma^i_{\inf}$ that, for any given control architecture and choice for ℓ, there exists a proper LTI controller $K(z)$ and a noise variance $\sigma_q^2 \in \mathbb{R}^+$ such that the resulting NCS is MSS and the channel SNR constraint is satisfied if and only if

$$\gamma^i_{\inf} < \Gamma. \tag{6}$$

Thus, by solving Problem 1, one will be able to find the choice of ℓ which yields the lowest stabilization requirements for the non transparent channel.

The second problem to be addressed is stated next:

Problem 2. Consider the setup and assumptions of Problem 1. Find, for any $\Gamma > \gamma^i_{\inf}$,

$$\left[\sigma_y^2\right]^i_\Gamma \triangleq \inf_{\substack{K\in S_i \\ \sigma_q^2 \in \mathbb{R}^+ \\ \gamma \le \Gamma}} \sigma_y^2, \tag{7}$$

where σ_y^2 is the stationary variance of the plant output. ■

By solving Problem 2, one will find the best performance (as measured by the plant output variance) that can be achieved in any of the considered architectures, when a specific choice of ℓ is made, while ensuring MSS and the satisfaction of the channel SNR constraint.

To solve Problems 1 and 2, we will first focus on a generalized feedback loop. Then, we will particularize these general results to Cases 1, 2 and 3.

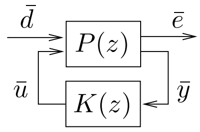

Fig. 2. Generalized Control Loop

5. MEAN SQUARE STABILITY

In this section we focus on Problem 1. We will start by considering the general setup depicted in Figure 2. In that figure, $P(z)$ is a generalized LTI plant, $K(z)$ is an LTI controller, and the signals $\bar{d}$, $\bar{e}$, $\bar{y}$ and $\bar{u}$ are, respectively, a vector that contains the external inputs, a signal related to closed loop performance, the controller input, and the controller output. We partition $P(z)$ as follows:

$$\begin{bmatrix} \bar{e} \\ \bar{y} \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} P_{11}(z) & P_{12}(z) \\ \hline P_{21}(z) & P_{22}(z) \end{array}\right]}_{P(z)} \begin{bmatrix} \bar{d} \\ \bar{u} \end{bmatrix}. \tag{8}$$

Assume that $P(z) \in \mathcal{R}_p$, and that $P_{22}(z) \in \mathcal{R}_{sp}$. Consider a doubly coprime factorization of $P_{22}(z)$, i.e., consider coprime factors N_i, D_i, N_d and D_d in $\mathcal{RH}_\infty$, with D_i and D_d biproper, such that $P_{22}(z) = N_d D_d^{-1} = D_i^{-1} N_i$, and

$$\begin{bmatrix} X_i & -Y_i \\ -N_i & D_i \end{bmatrix} \begin{bmatrix} D_d & Y_d \\ N_d & X_d \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \tag{9}$$

for some X_i, Y_i, X_d and Y_d in $\mathcal{RH}_\infty$, with X_i, X_d biproper. It is known (Zhou et al. (1996)) that all controllers $K(z) \in \mathcal{R}_p$ that make the feedback loop of Figure 2 internally stable and well posed are given by

$$K(z) = (X_i - QN_i)^{-1}(Y_i - QD_i), \tag{10}$$

where $Q(z)$ is a free parameter in $\mathcal{RH}_\infty$.

In the architectures considered in this paper, the generalized plant $P(z)$ can be found by setting

$$\bar{u} = u, \quad \bar{d} = \begin{bmatrix} d^T & q \end{bmatrix}^T, \quad \bar{y} = \begin{cases} x & \text{for Case 1,} \\ y & \text{for Case 2,} \\ \begin{bmatrix} y \\ \tilde{w} \end{bmatrix} & \text{for Case 3.} \end{cases} \tag{11a}$$

Given the definition of γ in (3), and since our objective in this section is mere stabilization, we set

$$\bar{e} = v. \tag{11b}$$

With the previous definitions, it is easy to see that $P(z)$ is such that (recall that $v = u_\ell$ and $w = \tilde{u}_\ell$)

$$\begin{bmatrix} v \\ \bar{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \eta_\ell^T \\ W(z) & P_{22}(z)\eta_\ell & P_{22}(z) \end{bmatrix} \begin{bmatrix} d \\ q \\ u \end{bmatrix}, \tag{12}$$

where

$$P_{22}(z) = \overline{G}(z) \text{ and } W(z) = \overline{G}(z) \text{ for Case 1,} \tag{13a}$$

$$P_{22}(z) = G(z) \text{ and } W(z) = G(z) \text{ for Case 2,} \tag{13b}$$

$$P_{22}(z) = \begin{bmatrix} G(z) \\ z^{-h}\eta_\ell^T \end{bmatrix} \text{ and } W(z) = \begin{bmatrix} G(z) \\ 0 \end{bmatrix} \text{ for Case 3.} \tag{13c}$$

We now introduce a definition necessary to derive our results. Consider a plant $P_{22}(z)$ with minimal state space representation given by $P_{22}(z) = (A, B, C, 0)$. Given assumptions of $G(z)$ in Problem 1, the poles of $P_{22}(z)$ are all simple. Denote by (A_J, B_J, C_J, D_J) the state space representation of $P_{22}(z)$ based on the Jordan representation of the matrix A above. Without loss of generality we assume that the j^{th} eigenvalue of A, say λ_j, is located in the j^{th} row of A_J.

***Definition** 2.* Assume that λ_j is a pole of $P_{22}(z)$. We say that λ_j is an unstable pole exclusively stabilizable by u_ℓ if $\lambda_j > 1$ and the j^{th} row of B_J has the form $B_{J_{j*}} = \alpha_j \eta_\ell^T$, where $\alpha_j \in \mathbb{C} - \{0\}$. ■

***Remark** 2.* Assuming that $G(z)$ has only simple poles, then $P_{22}(z)$ has only simple poles too for any choice of $P_{22}(z)$ in (13). To deal with conjugated complex poles, Definition 2 is not restricted to use a real Jordan representation of A, thus, using a complex Jordan representation we have that A_J is a diagonal matrix given by $A_J = diag\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ with $\lambda_j \in \mathbb{C}$. ■

The following theorem introduce some partial results required to solve Problem 1.

***Theorem** 1.* Consider a plant $P_{22}(z)$ with minimal state space representation given by $P_{22}(z) = (A, B, C, 0)$ that admit a coprime factorization given by $P_{22} = N_d D_d^{-1} = D_i^{-1} N_i$. Fix $\ell \in \{1, 2\}$. Then

(1) The set of finite NMP zeros of $D_{d_{\ell*}}$ is the set of unstable poles of P_{22} that are exclusively stabilizable by u_ℓ.

(2) $P_{22_{*\ell}}$ and $N_{i_{*\ell}}$ share their finite NMP zeros and relative degree.

■

Proof.

(1) It is known that D_d can be obtained from the state space representation of P_{22} as $D_d = (A + BF, B, F, I)$, where F is a matrix such that every eigenvalue of $A + BF$ is inside of the unit circle. Thus, its clear that $D_{d_{\ell*}} = (A + BF, B, \eta_\ell^T F, \eta_\ell^T)$. Now, by definition, an invariant zero of the state space representation of $D_{d_{\ell*}}$ is a complex number z_o such that the matrix

$$\begin{bmatrix} A + BF - z_o I & B \\ \eta_\ell^T F & \eta_\ell^T \end{bmatrix} \tag{14}$$

looses rank. It is worth noting that

$$rank\left(\begin{bmatrix} A + BF - z_o I & B \\ \eta_\ell^T F & \eta_\ell^T \end{bmatrix}\right) = rank\left(\begin{bmatrix} A - z_o I & B \\ 0 & \eta_\ell^T \end{bmatrix}\begin{bmatrix} I & 0 \\ F & I \end{bmatrix}\right) = row - rank\left(\begin{bmatrix} A - z_o I & B \\ 0 & \eta_\ell^T \end{bmatrix}\right). \tag{15}$$

Using the state space representation based on the Jordan form of A, it is easy to see that $z_o = \lambda_j$ makes the matrix

$$\begin{bmatrix} A_J - z_o I & B_J \\ 0 & \eta_\ell^T \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \lambda_1 - z_o & 0 & 0 & 0 \\ 0 & \lambda_2 - z_o & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n - z_o \end{bmatrix} & \begin{bmatrix} B_{J_{1*}} \\ B_{J_{2*}} \\ \vdots \\ B_{J_{n*}} \end{bmatrix} \\ 0 & \eta_\ell^T \end{bmatrix}. \tag{16}$$

loss rank if and only if $B_{J_{j*}} = \alpha_j \eta_\ell^T$, with $\alpha_j \in \mathbb{C} - \{0\}$. If the invariant zero z_o is minimum phase, then it can be cancelled by a pole of $D_{d_{\ell*}}$ (recall that $A + BF$ determine the poles of $D_{d_{\ell*}}$), hence minimum phase invariant zeros may not be a transmission zero of $D_{d_{\ell*}}$. On the other hand, a NMP invariant zero can not be cancelled by a pole of $D_{d_{\ell*}}$, furthermore, since $(A + BF, B)$ is stabilizable (also is $(A_J + B_J F, B_J)$), then the NMP invariant zeros described above are transmission zeros of $D_{d_{\ell*}}$ too. Hence the set of NMP zeros of $D_{d_{\ell*}}$ correspond to the set of unstable poles of P_{22} that are exclusively stabilizable by u_ℓ.

(2) It is known that N_i can be obtained from the state space representation of P_{22} as $N_i = (A + LC, B, C, 0)$, where L is a matrix such that every eigenvalue of $A + LC$ is inside of the unit circle. Thus, $N_{i_{*\ell}} = (A + LC, B\eta_\ell, C, 0)$, and an invariant zero z_o of such representation is such that the matrix

$$\begin{bmatrix} A + LC - z_o I & B\eta_\ell \\ C & 0 \end{bmatrix} \tag{17}$$

losses rank. Notice that

$$rank\left(\begin{bmatrix} A + LC - z_o I & B\eta_\ell \\ C & 0 \end{bmatrix}\right) = rank\left(\begin{bmatrix} I & L \\ 0 & I \end{bmatrix}\begin{bmatrix} A - z_o I & B\eta_\ell \\ C & 0 \end{bmatrix}\right) = column{-}rank\left(\begin{bmatrix} A - z_o I & B\eta_\ell \\ C & 0 \end{bmatrix}\right). \tag{18}$$

This implies that the invariant zeros obtained from (18) are also the invariant zeros of the state space representation $(A, B\eta_\ell, C, 0)$ (see the last term in (18)). Since $P_{22_{*\ell}} = (A, B\eta_\ell, C, 0)$, we conclude that every finite NMP transmission zero obtained from (18) is a finite NMP transmission zero of both $P_{22_{*\ell}}$ and $N_{i_{*\ell}}$.

On the other hand, the number of zeros at infinity corresponds to the relative degree of $P_{22_{*\ell}}$, say r, and is such that $P_{22_{*\ell}} z^r$ is left invertible when $z \to \infty$. Then, by noting that $P_{22_{*\ell}} = D_i^{-1} N_{i_{*\ell}}$, and given that D_i is biproper and invertible, it is clear that $P_{22_{*\ell}} z^r$ is left invertible when $z \to \infty$ if and only if $N_{i_{*\ell}} z^r$ is left invertible when $z \to \infty$. Hence, $N_{i_{*\ell}}$ has relative degree r too. This fact completes the proof.

■

The following theorem, based upon results by Braslavsky et al. (2007) and Rojas (2009), will enable us to provide a solution to Problem 1:

Theorem 2. Fix $\ell \in \{1, 2\}$. Consider the feedback loop of Figure 2, where $\bar{u}$, $\bar{d}$, $\bar{y}$, $\bar{e}$ and $P(z)$ are given by (11) and (12). Also consider the setup of Problem 1 and denote by $\mathcal{P}_\ell \triangleq \{p_1, p_2, \cdots, p_{n_p}\}$ the set of unstable poles of P_{22} that are exclusively stabilizable by u_ℓ, by $\mathcal{C}_\ell \triangleq \{c_1, c_2, \cdots, c_{n_c}\}$ the set of finite NMP zeros of the ℓ^{th} column of P_{22}, i.e., of $P_{22_{*\ell}}$, and by r the relative degree of $P_{22_{*\ell}}$. Then

$$\gamma_{\inf} \triangleq \inf_{\substack{K(z) \in S \\ \sigma_q^2 \in \mathbb{R}^+}} \frac{\sigma_v^2}{\sigma_q^2} = \gamma_p + \gamma_c + \gamma_r + \gamma_{rc}, \tag{19}$$

where S denote the set of all proper LTI controllers $K(z)$ that stabilize the loop of Figure 2 and

$$\gamma_p = \left(\prod_{i=1}^{n_p} |p_i|^2\right) - 1, \quad \gamma_c = \sum_{i=1}^{n_c}\sum_{j=1}^{n_c} \frac{R_{c_i}\bar{R}_{c_j}}{(c_i\bar{c}_j - 1)}, \quad \gamma_r = \sum_{k=1}^{r-1} |\beta_k|^2, \quad \gamma_{rc} = \sum_{j=1}^{r-1}\left|\sum_{i=1}^{n_c} R_{c_i}\ c_i^{j-1}\right|^2,$$
$$R_{c_i} = (1 - |c_i|^2)\prod_{\substack{j=1 \\ j \neq i}}^{n_c}\left(\frac{1 - c_i\bar{c}_j}{c_i - c_j}\right)\left(\xi_p(c_i) - \sum_{k=0}^{r-1} \beta_k c_i^{-k}\right), \tag{20}$$

where $\xi_p(z)$ and $\{\beta_k; k \in \mathbb{N}_0\}$ are such that

$$\xi_p(z) \triangleq \prod_{i=1}^{n_p} \left(\frac{1-\bar{p}_i z}{z-p_i} \right) = \sum_{k=0}^{\infty} \beta_k z^{-k}. \tag{21}$$

■

Proof. Denote by T_{xy} the closed loop transfer function from any signal x to another signal y in Figure 2. It is known that every transfer $T_{\bar{d}\bar{e}}$ associated to a controller in S can be written in terms of the partitioned generalized plant $P(z)$ and the coprime factorization of $P_{22}(z)$ as

$$T_{\bar{d}\bar{e}} \triangleq T^o_{\bar{d}\bar{e}} - P_{12} D_d Q D_i P_{21}, \tag{22}$$

where $T^o_{\bar{d}\bar{e}} \triangleq P_{11} + P_{12} D_d Y_i P_{21}$. Note that $T^o_{\bar{d}\bar{e}}$, $P_{12} D_d$, and $D_i P_{21}$ are all in $\mathcal{RH}_\infty$.

The variance of v can be written as

$$\sigma_v^2 = ||T_{\bar{d}v} \Omega_{\bar{d}}||_2^2\,, \tag{23}$$

where

$$\Omega_{\bar{d}} = \text{diag}\{\Omega_d, \sigma_q\} \tag{24}$$

and $T_{\bar{d}v}$ can be obtained in terms of Q by using (22). Then, considering (23) above and the definition of γ in (3), is easy to see that

$$\gamma = \left|\left| T_{\bar{d}v} \Omega_{\bar{d}} \frac{1}{\sigma_q} \right|\right|_2^2 \geq ||T_{qv}||_2^2\,, \tag{25}$$

where the gap in (25) can be made arbitrarily small by choosing $\sigma_q^2 \to \infty$. From (22) and (12) we have

$$||T_{qv}||_2^2 = ||D_{d_{\ell *}} Y_i P_{22_{*\ell}} - D_{d_{\ell *}} Q D_i P_{22_{*\ell}}||_2^2\,. \tag{26}$$

Since (9) holds and $r \geq 1$, we have that

$$D_{d_{\ell *}} Y_i P_{22_{*\ell}} = (D_{d_{\ell *}} X_{i_{*\ell}} - 1) \in \mathcal{RH}_2. \tag{27}$$

From (27) and considering that $N_{i_{*\ell}} = D_i P_{22_{*\ell}}$ we write (26) as

$$||T_{qv}||_2^2 = ||D_{d_{\ell *}} X_{i_{*\ell}} - 1 - D_{d_{\ell *}} Q N_{i_{*\ell}}||_2^2\,. \tag{28}$$

Theorem 1 implies that the set of finite NMP zeros of $D_{d_{\ell *}}$ correspond to $\mathcal{P}_\ell$, then by using (28), and since $\xi_p(z)$ is unitary and $\mathcal{RH}_2$ is orthogonal with $\mathcal{RH}_2^\perp$, it is easy to show that

$$||T_{qv}||_2^2 = \gamma_p + ||\xi_p(z) D_{d_{\ell *}} X_{i_{*\ell}} - \xi_p(\infty) - \xi_p(z) D_{d_{\ell *}} Q N_{i_{*\ell}}||_2^2 \tag{29}$$

where

$$\gamma_p \triangleq ||\xi_p(z) - \xi_p(\infty)||_2^2 = \left(\prod_{i=1}^{n_p} |p_i|^2 \right) - 1 \tag{30}$$

follows from the residue theorem.

Considering (27), the expansion in (21), and defining

$$\Phi_o \triangleq \left(\xi_p(z) D_{d_{\ell *}} Y_i P_{22_{*\ell}} + \sum_{k=r}^{\infty} \beta_k z^{-k} \right) \in \mathcal{RH}_2, \tag{31}$$

we can use orthogonality properties again to write

$$||T_{qv}||_2^2 = \gamma_p + \gamma_r + ||\Phi_o - \xi_p(z) D_{d_{\ell *}} Q N_{i_{*\ell}}||_2^2\,, \tag{32}$$

where

$$\gamma_r \triangleq \left|\left| \sum_{k=0}^{r-1} \beta_k z^{-k} - \xi_p(\infty) \right|\right|_2^2 = \sum_{k=1}^{r-1} |\beta_k|^2 \tag{33}$$

From theorem 1 we have that the set of finite NMP zeros of $N_{i_{*\ell}}$ corresponds to $\mathcal{C}_\ell$. To deal with those NMP zeros in (32) define

$$\xi_c(z) \triangleq \prod_{j=1}^{n_c} \left(\frac{1-\bar{c}_i z}{z-c_i} \right), \quad \Phi \triangleq \Phi_o \xi_c(z) = [\Phi]_{\mathcal{H}_2} + [\Phi]_{\mathcal{H}_2^\perp}\,. \tag{34}$$

Notice that the set of unstable poles of Φ corresponds to $\mathcal{C}_\ell$. Hence $[\Phi]_{\mathcal{H}_2^\perp}$ has the form

$$[\Phi]_{\mathcal{H}_2^\perp} = \sum_{i=1}^{n_c} \frac{R_{c_i}}{(z-c_i)}, \tag{35}$$

where R_{c_i} is the residue of Φ at the pole c_i. That residue can be calculated as $R_{c_i} = [(z-c_i)\Phi]|_{z=c_i}$. Then by using (34), (31), and (21), it is easy to see that

$$R_{c_i} = (1-|c_i|^2) \prod_{\substack{j=1\\ j\neq i}}^{n_c} \left(\frac{1-c_i\bar{c}_j}{c_i-c_j}\right) \left(\xi_p(c_i) - \sum_{k=0}^{r-1} \beta_k c_i^{-k}\right). \tag{36}$$

Thus, (32), (34), and 2-norm properties yield

$$||T_{qv}||_2^2 = \gamma_p + \gamma_r + \gamma_c + \left|\left|[\Phi]_{\mathcal{H}_2} - \xi_p(z) D_{d_{\ell *}} Q N_{i_{*\ell}} \xi_c(z)\right|\right|_2^2, \tag{37}$$

where

$$\begin{aligned}
\gamma_c &\triangleq \left|\left|[\Phi]_{\mathcal{H}_2^\perp}\right|\right|_2^2 = \left|\left|\sum_{i=1}^{n_c} \frac{R_{c_i}}{(z-c_i)}\right|\right|_2^2 \\
&= \frac{1}{2\pi j} \oint \left(\sum_{i=1}^{n_c} \frac{R_{c_i}}{(z-c_i)}\right) \left(\sum_{j=1}^{n_c} \frac{\bar{R}_{c_j}}{(1-\bar{c}_j z)}\right) dz \\
&= \frac{1}{2\pi j} \oint \sum_{i=1}^{n_c} \sum_{j=1}^{n_c} \frac{R_{c_i}\bar{R}_{c_j}}{(z-c_i)(1-\bar{c}_j z)} dz \\
&= \sum_{i=1}^{n_c} \sum_{j=1}^{n_c} \frac{R_{c_i}\bar{R}_{c_j}}{(c_i\bar{c}_j - 1)}
\end{aligned} \tag{38}$$

In (37), $N_{i_{*\ell}}$ has relative degree r, but $[\Phi]_{\mathcal{H}_2}$ may have relative degree 1. Hence, may there exist no $Q \in \mathcal{RH}_\infty$ such that the last term in (37) is zero. Define

$$[\Delta]_{\mathcal{H}_2^\perp} \triangleq \sum_{i=1}^{n_c} R_{c_i} \left(\sum_{j=1}^{r-1} c_i^{j-1} z^{-j}\right) z^{r-1}, \tag{39}$$

$$[\Delta]_{\mathcal{H}_2} \triangleq \sum_{i=1}^{n_c} R_{c_i} \left(\sum_{j=r}^{\infty} c_i^{j-1} z^{-j}\right) z^{r-1}. \tag{40}$$

Note that (39) and (40) are such that

$$[\Phi]_{\mathcal{H}_2^\perp} = \left([\Delta]_{\mathcal{H}_2^\perp} + [\Delta]_{\mathcal{H}_2}\right) z^{-(r-1)}, \tag{41}$$

and, since $[\Phi]_{\mathcal{H}_2} = \Phi - [\Phi]_{\mathcal{H}_2^\perp}$ and (37) holds, we can write

$$||T_{qv}||_2^2 = \gamma_p + \gamma_r + \gamma_c + \gamma_{rc} + \left|\left|\Phi - [\Delta]_{\mathcal{H}_2} z^{-(r-1)} - \xi_p(z) D_{d_{\ell *}} Q N_{i_{*\ell}} \xi_c(z)\right|\right|_2^2, \tag{42}$$

where

$$\begin{aligned}
\gamma_{rc} &\triangleq \left|\left|[\Delta]_{\mathcal{H}_2^\perp}\right|\right|_2^2 = \left|\left|\sum_{i=1}^{n_c} R_{c_i} \left(\sum_{j=1}^{r-1} c_i^{j-1} z^{-j}\right)\right|\right|_2^2 \\
&= \sum_{j=1}^{r-1} \left|\sum_{i=1}^{n_c} R_{c_i}\ c_i^{j-1}\right|^2.
\end{aligned} \tag{43}$$

Since $D_{d_{\ell *}}$ is right-invertible and $N_{i_{*\ell}}$ is left-invertible, we can choose

$$Q(z) = \left(\xi_p^{-1} D_{d_{\ell *}}^\dagger \left(\Phi - [\Delta]_{\mathcal{H}_2} z^{-(r-1)}\right) N_{i_{*\ell}}^\dagger \xi_c^{-1}\right) \in \mathcal{RH}_\infty \tag{44}$$

such that

$$\gamma_{\inf} \triangleq \inf_{Q \in \mathcal{RH}_\infty} ||T_{qv}||_2^2 = \gamma_p + \gamma_r + \gamma_c + \gamma_{rc} \tag{45}$$

This completes the proof. ■

***Remark** 3.* It is easy to see that, if $\mathcal{C}_\ell = \phi$ (i.e. $n_c = 0$), then $\gamma_c = \gamma_{rc} = 0$. Also, if $r = 1$, then $\gamma_r = \gamma_{rc} = 0$ and, if $\mathcal{P}_\ell = \phi$ (i.e. $n_p = 0$), then $\gamma_{\inf} = 0$. ■

We now use Theorem 2 to solve Problem 1:

***Corollary** 1.* Consider Problem 1. Then:

(1) For the architecture of Case 1, the minimal SNR compatible with MSS is given by

$$\gamma_{\inf}^1 = \gamma_p, \tag{46}$$

where γ_p is defined as in Theorem 2 with $P_{22}(z) = \overline{G}(z)$ (see (13)).

(2) For the architecture of Case 2, the minimal SNR compatible with MSS is given by

$$\gamma_{\inf}^2 = \gamma_{\inf}^1 + \gamma_c + \gamma_r + \gamma_{rc}, \tag{47}$$

where γ_c, γ_r and γ_{rc} are defined as in Theorem 2 with $P_{22}(z) = G(z)$.

(3) For the architecture of Case 3, the minimal SNR compatible with MSS is given by

$$\gamma_{\inf}^3 = \gamma_{\inf}^1 + \gamma_r, \tag{48}$$

where γ_r is defined as in Theorem 2 with $P_{22}(z) = [G(z)^T \quad z^{-h}\eta_\ell]^T$. In this case, $r = \min\{\bar{r}, h\}$, where $\bar{r}$ is the relative degree of $G(z)_{*\ell}$, and h is the delay of the feedback channel. ■

Proof. First we prove that, given a plant $G(z)$, the set $\mathcal{P}_\ell$ is the same for Cases 1, 2 and 3. Consider a right coprime factorization of $G(z) = (A_G, B_G, C_G, 0)$ and $P_{22}(z) = (A, B, C, 0)$ given respectively by $G(z) = \bar{N}_d \bar{D}_d^{-1}$ and $P_{22}(z) = N_d D_d^{-1}$. Recall from the proof of Theorem 1 that D_d can be obtained from the state space representation of P_{22} as $D_d = (A + BF, B, F, I)$. Then

- For Case 1 we have that $P_{22}(z) = \overline{G}(z) = (A_G, B_G, I, 0)$ and hence $D_d = (A_G + B_G F, B_G, F, I)$.
- For Case 2 we have that $P_{22}(z) = G(z) = (A_G, B_G, C_G, 0)$ and hence $D_d = \bar{D}_d = (A_G + B_G F, B_G, F, I)$.
- For Case 3 we have that $P_{22}(z) = \begin{bmatrix} G(z) \\ z^{-h}\eta_\ell^T \end{bmatrix}$ and hence we can construct N_d and D_d as $N_d = \begin{bmatrix} \bar{N}_d \\ z^{-h}\eta_\ell^T \bar{D}_d \end{bmatrix}$ and $D_d = \bar{D}_d$.

Since $D_d = \bar{D}_d$ remains equal for Cases 1, 2 and 3, then the set $\mathcal{P}_\ell$ is the same for the three Cases, and correspond to the set of unstable poles of $G(z)$ exclusively stabilizable by u_ℓ. This fact allows one to prove Corollary 1 as follows:

(1) Since $\overline{G}(z) = (A_G, B_G, I, 0)$, the invariant zeros of that realization are the values of z such that the matrix

$$\begin{bmatrix} A_G - zI & B_G \\ I & 0 \end{bmatrix} \tag{49}$$

losses rank. However since (A_G, B_G) is stabilizable, the matrix $\begin{bmatrix} A_G - zI & B_G \end{bmatrix}$ is full row rank, furthermore the matrix $\begin{bmatrix} A_G^T - zI & I \end{bmatrix}^T$ is full column rank. Thus, there is no value of z corresponding an invariant zero of such representation. This implies that $\overline{G}(z)$ has no finite zeros. Now, we show that $\overline{G}(z)_{*\ell}$ has relative degree one by proving that $z\overline{G}(z)_{*\ell}$ is left invertible when $z \to \infty$. Then, from $\overline{G}(z)_{*\ell} = (A_G, B_G\eta_\ell, I, 0)$ and by using invertibility properties of matrices we can write:

$$\begin{aligned} z\overline{G}(z)_{*\ell} &= z(zI - A_G)^{-1} B_G \eta_\ell && (50) \\ &= (I - z^{-1}A_G)^{-1} B_G \eta_\ell && (51) \\ &= (I + (I - z^{-1}A_G)^{-1} z^{-1} A_G) B_G \eta_\ell && (52) \\ &= B_G \eta_\ell + (zI - A_G)^{-1} A_G B_G \eta_\ell && (53) \end{aligned}$$

Without lost of generality we assume that $B_G\eta_\ell$ is full column rank and therefore $z\overline{G}(z)_{*\ell}$ is left-invertible when $z \to \infty$. Hence, $\overline{G}(z)_{*\ell}$ has relative degree one.
Thus, since $\overline{G}(z)$ has the same unstable poles of $G(z)$, by using (13) and Theorem 2 with $n_c = 0$ and $r = 1$, we obtain (46).

(2) Use Theorem 2 with $P_{22}(z) = G(z)$.

(3) In this case, $P_{22}(z) = \begin{bmatrix} G(z)^T & \eta_\ell z^{-h} \end{bmatrix}^T$, and thus $P_{22_{*\ell}}$ has no finite zeros (equivalently $n_c = 0$) and $r = \min\{\bar{r}, h\}$ zeros at infinity. Hence, the proof is completed upon using Theorem 2. ■

As expected from previous results by Braslavsky et al. (2007) and Silva et al. (2010), Corollary 1 shows that the presence of unstable poles limit the minimal SNR compatible with MSS in all the considered architectures. In the present case, however, only these unstable poles exclusively stabilizable by u_ℓ play a role. This is consistent with intuition. In the state feedback case (Case 1), the unstable poles of $G(z)$ exclusively stabilizable by u_ℓ are the only source of limitation, whereas in the output feedback case (Case 2), the NMP zeros and relative degree of $P_{22_{*\ell}} = G(z)_{*\ell}$ also play a relevant role. When feedback channel is used in the output feedback case (Case 3) the effects of NMP zeros are annulled, while the effect of relative degree depends on the feedback channel delay h. Indeed, if $h = 1$ in the latter case, then the minimal SNR for stability is just as in the state feedback case.

6. OPTIMAL CONTROLLER DESIGN

The results of Section 5 allow one to establish necessary and sufficient conditions on the maximum admissible channel SNR Γ so as to guarantee MSS. As such those conditions are also necessary and sufficient for Problem 2 to be feasible. In this section, we review two approaches to solve Problem 2. The first one allows one to deal with Cases 1, 2 or 3, while the second approach is tailored to the state feedback case.

6.1 The general case

As in Section 5, we will consider the general setup of Figure 2, where $\bar{u}$, $\bar{d}$ and $\bar{y}$ are given by (11a). Since in the present case we are interested in both the output variance σ_y^2 and the channel input variance σ_v^2, we set

$$\bar{e} = \begin{bmatrix} y^T & v \end{bmatrix}^T. \tag{54}$$

Thus, $P(z)$ becomes such that

$$\begin{bmatrix} y \\ v \\ \bar{y} \end{bmatrix} = \begin{bmatrix} G(z) & G(z)\eta_\ell & G(z) \\ 0 & 0 & \eta_\ell^T \\ W(z) & P_{22}(z)\eta_\ell & P_{22}(z) \end{bmatrix} \begin{bmatrix} d \\ q \\ u \end{bmatrix} \tag{55}$$

where $P_{22}(z)$ and $W(z)$ are as in (13).

Irrespective of the case under study (Cases 1, 2 or 3), it is possible to use the Youla-Kucera parametrization to write the plant output variance and the channel input variance as convex functions of the parameter $Q(z) \in \mathcal{RH}_\infty$. We introduce the notation $J^i_{\sigma_q^2}(Q)$ to refer to the output variance in Case i, as a convex function of the parameter $Q(z)$ and considering σ_q^2 as a parameter. The notation $R^i_{\sigma_q^2}(Q)$ is used to denote the corresponding channel input variance. It thus follows immediately that

$$\left[\sigma_y^2\right]^i_\Gamma = \inf_{\sigma_q^2 \in \mathbb{R}^+} \inf_{\substack{Q \in \mathcal{RH}_\infty \\ R^i_{\sigma_q^2}(Q) \le \Gamma\sigma_q^2}} J^i_{\sigma_q^2}(Q). \tag{56}$$

For any fixed σ_q^2, the inner optimization problem in (56) is a standard quadratic optimal control problem and can be addressed using standard tools (Zhou et al. (1996); Boyd et al. (1994)). The outer problem in (56) corresponds to a line search and, in principle, should pose no numerical difficulties. Then, by solving

(56), the optimal controller is obtained from (10). A detailed study of the nested problem in (56) can be found in Silva et al. (2010).

6.2 The state feedback case

In the state feedback case, one can focus, without loss of generality, on static controllers (Rotea and Khargonekar (1991)). The following result presents a solution to Problem 2 for this case, when d is white:

***Theorem** 3.* Consider Problem 2 and assume d to be white. Define the following optimization problem in the matrix variables (of appropriate dimensions) X, Z, Λ and σ^2:

$$\text{Find: } \rho \triangleq \inf \ \text{trace}\{\Lambda\} \tag{57}$$

$$\text{s.t.: } X > 0, \quad \Lambda \geq 0, \quad \sigma^2 \geq 0 \tag{58}$$

$$\Lambda - C_G X C_G^T > 0 \tag{59}$$

$$\begin{bmatrix} X - B_G P_d B_G^T - B_G \eta_\ell \eta_\ell^T B_G^T \sigma^2 & A_G X + B_G Z \\ \star & X \end{bmatrix} > 0 \tag{60}$$

$$\begin{bmatrix} \Gamma\sigma^2 & \eta_\ell^T Z \\ \star & X \end{bmatrix} > 0, \tag{61}$$

where $\star$ represent the entries that make each matrix symmetric, P_d is the covariance matrix of d, $B_G \triangleq [B_1 \ \ B_2]$, and all the remaining symbols are as in the description of Case 1. If $\Gamma > \gamma_{\text{inf}}$, then:

(1) The optimization problem in (57)–(61) is feasible and $\left[\sigma_y^2\right]_\Gamma^1 = \rho$.
(2) If $Z_{\text{opt}}, X_{\text{opt}}$ and σ^2_{opt} are the optimal values of Z, X and σ^2, then the choice of the controller $K(z) = Z_{\text{opt}} X_{\text{opt}}^{-1}$ and noise variance $\sigma_q^2 = \sigma^2_{\text{opt}}$ is such that the NCS considered in Case 1 is MSS, the channel SNR constraint is satisfied, and $\sigma_y^2 = \left[\sigma_y^2\right]_\Gamma^1$. ■

Proof. The result follows upon adapting Theorem 1 in Pulgar et al. (2011). ■

Theorem 3 presents a characterization of the solution to Problem 2 in terms of the solution to a convex problem involving LMIs (Boyd et al. (1994)). As such, the proposed approach is computationally attractive. We note that, by solving the LMI-based optimization problem in (57)–(61), one gets not only a characterization of the best achievable performance for a given channel SNR constraint, but also a one-shot characterization of the controller and the channel noise variance that allows one to achieve such performance.

7. AN EXAMPLE

In this section, we illustrate the results of this paper for each of the considered control architectures. For Case 3, we use $h = 1$ and $h = 2$, and we use the notation $\gamma^3_{\text{inf}_h}$ to make explicit the value used for h.

Consider a plant $G(z)$ as in (1), where

$$A_G = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 1.5 & 0.5 & 1 \\ 0.2 & 0.3 & 1.8 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad B_G = \begin{bmatrix} B_1 & B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad D_G = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If $\ell = 1$ then $\mathcal{P}_1 = \{3\}$, $\mathcal{C}_1 = \phi$, and $r = 1$. Since $G(z)_{*1}$ has no NMP zeros, and the relative degree is one, the minimal SNR in all cases is equal, namely $\gamma^1_{\text{inf}} = \gamma^2_{\text{inf}} = \gamma^3_{\text{inf}_1} = \gamma^3_{\text{inf}_2} = 8$. On the other hand, when

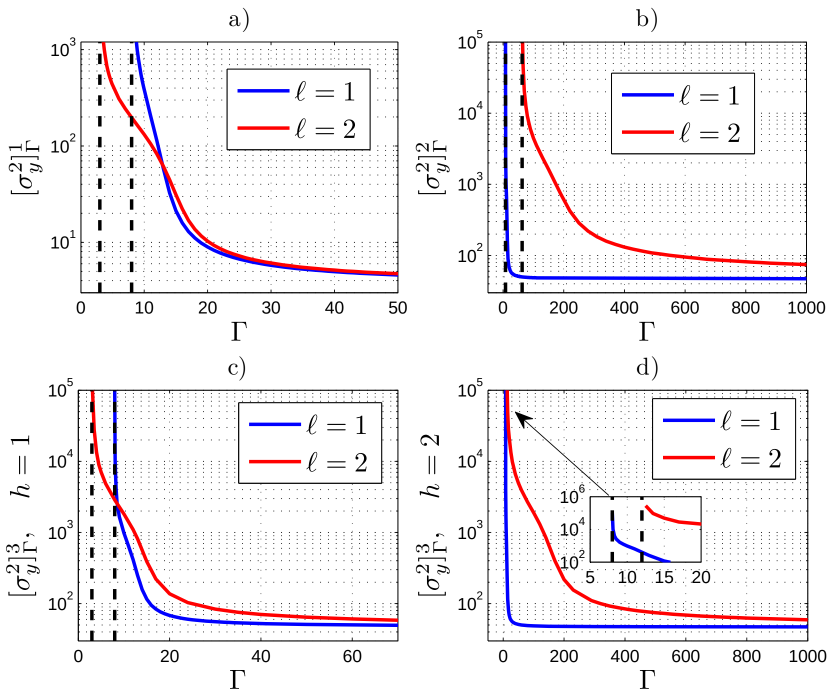

Fig. 3. Minimum output variance $\left[\sigma_y^2\right]_\Gamma^i$ as a function of the maximum admissible SNR Γ for $\ell = 1$ and $\ell = 2$: *a*) State feedback, *b*) Output feedback, *c*) Output feedback with feedback channel when $h = 1$, *d*) Output feedback with feedback channel when $h = 2$. (All curves are convex, but look otherwise in the log scale used here)

$\ell = 2$, we have $\mathcal{P}_2 = \{2\}$, $\mathcal{C}_2 = \{1.3\}$ and $r = 2$. In this case, there exists one NMP zero and the relative degree is larger than one. Hence, we have that $\gamma_{\text{inf}}^1 = 3$, $\gamma_{\text{inf}}^2 = 62.69$, $\gamma_{\text{inf}_1}^3 = 3$ and $\gamma_{\text{inf}_2}^3 = 12$.

Figure 3 shows the minimum variance $\left[\sigma_y^2\right]_\Gamma^i$ as a function of Γ for all architectures, with $\ell \in \{1, 2\}$ as a parameter. The vertical dashed lines denote the minimal SNR compatible with MSS calculated previously for each case. Figure 3.*a*) refers to the state feedback control scheme. Notice that we can stabilize the loop with smaller SNR if we choose $\ell = 2$. However as Γ increases, there is a point above which it is preferable to choose $\ell = 1$ to obtain a smaller output variance. Figure 3.*b*) shows the performance in the output feedback control architecture. In that case, it is clear that the choice $\ell = 1$ is the best one, from the stabilization and the performance points of view. Indeed, with $\ell = 2$ the requirements on the channel SNR increases considerably respect to $\ell = 1$. In Figure 3.*c*), we use the architecture of Case 3 with $h = 1$. We can see that the situation is as in Figure 3.*a*), i.e., the best performance is achieved with $\ell = 2$ for some values of Γ and with $\ell = 1$ for other values. Figure 3.*d*) also refers to Case 3 but, in this case, $h = 2$. The additional cost, arising from larger feedback channel delays, appears for $\ell = 2$.

The example illustrate an interesting issue: the choice of channels that demands the less requirements on the SNR channel for stabilization, is not necessarily the one that achieves the best performance (see Figures 3.*a*) and 3.*c*)). This implies that the choice of ℓ is not that trivial.

8. CONCLUSION

This paper has studied partly networked control architectures for TITO LTI plant models. We considered three architectures, including state and output feedback architectures, and an output feedback architectures with channel feedback. In each case, one of the control inputs has to be transmitted over an SNR constrained channel, whilst the remaining input is transmitted over a transparent channel. Our results show that the minimal SNR compatible with MSS depends on the particular architecture considered, and on the control input that is sent over the constrained channel. Performance issues have also been addressed in this paper by adapting results available in the literature. An interesting conclusion of our work is that the location of the SNR constrained channel that reduces the SNR requirements for stabilization, is not necessarily the one that allows one to achieve the best performance.

Future work should focus on networked situations involving several constrained channels. Also, the consideration of decentralized or distributed control architectures is an interesting subject for future research.

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