Optimal State-Feedback Design for MIMO systems subject to multiple SNR constraints ⋆

Sebastián A. Pulgar * **Eduardo I. Silva** * **Mario E. Salgado** *

* *Departamento de Electrónica, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile (e-mail: sebastian.pulgar@usm.com, eduardo.silva@usm.com, mario.salgado@usm.com).*

Abstract: This paper studies optimal control problems associated to networked systems. In particular, we focus on a case where a multiple-input multiple-output (MIMO) plant is controlled over a channel subject to multiple signal-to-noise ratio (SNR) constraints. For this setup, we establish necessary and sufficient condition for the existence of static state-feedback controllers that stabilize the plant in a mean square sense, while satisfying the channel SNR constraints. This characterization is given in terms of a convex optimization problem involving linear matrix inequalities (LMIs). We also provide a characterization of the best achievable performance in the setup considered, as well as for the state-feedback controller achieving that performance. As an application of our results, we study a networked situation where communication takes place over two erasures channels. To do so, we exploit a recently developed equivalence between networked control problems over erasure channels and SNR constrained optimal control problems. Finally, numerical examples are presented to illustrate the main results.

Keywords: Networked control systems, signal-to-noise ratio, optimal control, data dropouts.

1. INTRODUCTION

Networked control systems (NCSs) are control systems closed over constrained communication channels. The study of such systems has received much attention in the recent literature (see, e.g., Antsaklis and Baillieul (2007) and the references therein). Most of the work has focused on data-rate constraints (Nair et al. (2007)), data dropouts (Schenato et al. (2007)), random delays (Zhang et al. (2005)), and signal-to-noise ratio (SNR) constraints (Braslavsky et al. (2007); Rojas et al. (2008); Silva et al. (2010)). In this paper, we focus on MIMO plants controlled over an arbitrary number of (possible MIMO) additive noise channels subject to SNR constraints.

As a first step in the study of SNR constrained systems, Braslavsky et al. (2007) considered the stabilization of a single-input single-output plant closed over a scalar power constrained additive noise channel. The main results in Braslavsky et al. (2007) is a closed form characterization of the minimal channel SNR compatible with mean square stability. However, the results in Braslavsky et al. (2007) do not provide performance guarantees. Rojas et al. (2008) and Freudenberg et al. (2010), considered performance and robustness and related issues, as well as the problem of disturbance attenuation over SNR constrained additive noise channels. Further work was carried out by Silva et al. (2010), where a general LTI control architecture involving one scalar SNR constrained channel was studied. The works referred to above provide several insights, but consider situations where only one scalar channel is present. Thus the problem of control system design for MIMO plants controlled over multiple SNR constrained channels is still open.

The recent paper by Pulgar et al. (2010) addresses the problem of optimal static state-feedback control design for MIMO LTI plants controlled over two SNR constrained channels. To that end, Pulgar et al. (2010) first showed that the corresponding design problem is equivalent to the optimal design of mode-independent static state-feedback controllers for (a class of) Markov jump linear systems (MJLSs; Costa et al. (2005)). Unfortunately, the results available in MJLS theory do not provide a solution to that problem. Indeed, most results related to the optimal control of MJLSs focus on mode-dependent controllers, where it is assumed that the state of the underlying Markov chain is perfectly known at any time instant, and without delay (Costa et al. (2005); Geromel et al. (2009); Xiong and Lam (2007)). Results for the mode independent case are presented in Shu et al. (2010) and do Val et al. (2002). Shu et al. (2010) focus on stabilization using output feedback controllers, whereas do Val et al. (2002) present sufficient conditions for the existence of stabilizing state-feedback controllers, and provide an upper bound on the best achievable performance, when only partial information on the Markov chain state is available. By using the results in do Val et al. (2002), Pulgar et al. (2010) established an upper bound on the best achievable

⋆ The authors acknowledge the support from UTFSM and CONICYT through grants Anillo ACT-53, FONDECYT 3100024 and the Advanced Human Capital Program.

performance in the considered two channel control architecture, and also established sufficient conditions for the existence of stabilizing controllers.

As a first contribution of this paper, we provide a methodology to optimally design static state-feedback controllers for MIMO systems subject to multiple SNR constraints. The approach is based upon LMIs (see, e.g, Boyd et al. (1994)), and provides both necessary and sufficient conditions for the existence of optimal controllers of the considered class. By exploiting these results, and using the equivalence unveiled by Pulgar et al. (2010), we also solve an optimal static state-feedback control problem for NCSs closed over two analog erasure channels.

An interesting by-product of our results, and what we believe is a second contribution of this work, is the solution of a mode-independent optimal static state-feedback control problem for a restricted class of MJLS.

The remainder of this paper is organized as follows: Section 2 describes the problem setup. Section 3 presents the first contribution of this paper. Section 4 shows an application of our results to the solution to a problem left open by Pulgar et al. (2010) which involved NCSs closed over unreliable channels. Section 5 presents numerical examples, and conclusions are drawn in Section 6.

Notation: $\mathbb{R}$ and $\mathbb{N}$ refer to the real and natural numbers, respectively. $\mathbb{N}_0 \triangleq \mathbb{N} \cup \{0\}$ and $\mathbb{R}^+ \triangleq \{x \in \mathbb{R} : 0 < x < \infty\}$.$\mathcal{P}\{*\}$ stands for the probability of $(*)$ and $\mathcal{E}\{*\}$ denotes the expectation of $(*)$. Given a matrix W, W^T and W^H denote its transpose and conjugate transpose, respectively. $0_{n\times m}$ denotes the $n \times m$ zero matrix, I_n denotes the $n \times n$ identity matrix, and $0_n \triangleq 0\, I_n$. The notation $\text{diag}\{x_1, \cdots, x_n\}$, or simply $\text{diag}\{x_i\}$, refers to a block diagonal matrix with diagonal blocks given by x_i. If x is a wide sense stationary (wss) (or asymptotically wss) process, then P_x denotes its covariance matrix (or stationary covariance matrix) and $\sigma_x^2 \triangleq \text{trace}\{P_x\}$ its variance (or stationary variance). We say that a random variable (process) is a second order one if and only if it has finite mean and finite second order moments for all time instants $k \in \mathbb{N}_0$ (and also when $k \to \infty$). We use ρ for the forward shift operator.

2. PROBLEM SETUP

In this paper we focus on the NCS of Fig. 1, where G is a MIMO LTI system whose state x is available for measurement, K is a static state-feedback controller, u is the controller output, d is a disturbance, and z is a signal that reflects closed loop performance. The NCS of Fig. 1 also comprises a possibly multi-input multi-output additive noise channel, with input v, output w, and noise q.

We assume that G has the state space description

$$\begin{bmatrix} x(k+1) \\ \hdashline v(k) \\ z(k) \end{bmatrix} = \left[\begin{array}{c:ccc} A & B_u & B_d & B_w \\ \hdashline C_v & D_{uv} & D_{dv} & D_{wv} \\ C_z & D_{uz} & D_{dz} & D_{wz} \end{array}\right] \begin{bmatrix} x(k) \\ \hdashline u(k) \\ d(k) \\ w(k) \end{bmatrix}, \quad (1)$$

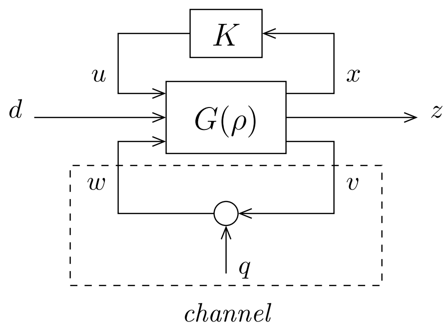

Fig. 1. Networked control system closed over an SNR constrained additive white noise channel.

with $k \in \mathbb{N}_0$, and where $x(k) \in \mathbb{R}^n$, $x(0) = x_0$, $u(k) \in \mathbb{R}^r$, $d(k) \in \mathbb{R}^p$, $w(k) \in \mathbb{R}^\ell$, $z(k) \in \mathbb{R}^s$, and $v(k) \in \mathbb{R}^\ell$.

We will work under the following assumptions:

***Assumption** 1.*

a) x_0 is a second order random variable.
b) The disturbance d is a zero mean second order white noise sequence, uncorrelated with x_0, and with covariance matrix $P_d \triangleq \Omega_d \Omega_d^H > 0$.
c) D_{wv} has a strictly lower (or upper) triangular structure. □

We note that Assumption 1c) guarantees that the system of Fig. 1 is well posed [1] for any choice of the controller K.

The channel of Fig. 1 is formally defined next:

***Definition** 1.* The channel of Fig. 1, with input v and output w, is an SNR constrained additive white noise channel if and only if, $\forall k \in \mathbb{N}_0, \forall v(k) \in \mathbb{R}^\ell$,

$$w(k) = q(k) + v(k), \quad q(k) \triangleq \begin{bmatrix} q_1(k)^T & \cdots & q_c(k)^T \end{bmatrix}^T, \quad (2)$$

where $q_i(k) \in \mathbb{R}^{n_i}$, $i \in \{1, \dots, c\}$, $\ell = n_1 + \cdots + n_c$, q_i is a zero mean white noise sequence uncorrelated with (x_0, d), and q_i is not correlated with q_j $\forall i \neq j$. In addition, the covariance matrix of q_i, i.e., P_{q_i}, is a design variable that can be chosen within the class of all $n_i \times n_i$ positive semidefinite matrices subject to the stationary SNR constraint

$$P_{v_i} \leq \Gamma_i P_{q_i}, \quad (3)$$

where P_{v_i} is the stationary variance of v_i and $\Gamma_i \in \mathbb{R}^+$ is the maximum admissible SNR for channel i. □

***Remark** 1.* Given Definition 1, P_q is block diagonal with diagonal blocks P_{q_i}. However, each P_{q_i} may be non diagonal. □

The partition of q in Definition 1 induces a corresponding partition on w and v, namely $w \triangleq \begin{bmatrix} w_1^T & \cdots & w_c^T \end{bmatrix}^T$ and $v \triangleq \begin{bmatrix} v_1^T & \cdots & v_c^T \end{bmatrix}^T$, with $w_i(k), v_i(k) \in \mathbb{R}^{n_i}$. This also implies a partition on the system matrices. Define

$$C_{v_i} = \eta_i C_v\ , \quad D_{uv_i} = \eta_i D_{uv} \quad \text{and} \quad D_{dv_i} = \eta_i D_{dv}, \quad (4)$$

where, $\forall i \in \{1, \dots, c\}$,

$$\eta_i \triangleq \begin{bmatrix} 0_{n_i \times (n_1 + \cdots + n_{i-1})} & I_{n_i} & 0_{n_i \times (n_{i+1} + \cdots + n_c)} \end{bmatrix}. \quad (5)$$

As discussed by Silva et al. (2010), SNR constrained additive white noise channels arise when standard input power constrained additive white (Gaussian) noise channels (see,

[1] In the standard sense defined in, e.g., Zhou et al. (1996)

e.g., Braslavsky et al. (2007); Cover and Thomas (2006)) are used with pre- and post-scaling factors. Choosing such scaling factors amounts to choosing the variance of the equivalent noise q. The ratio between the maximum admissible channel input power and the underlying channel noise variance defines the maximum admissible SNR of the equivalent SNR constrained channel.

As foreshadowed before, we are interested in static state-feedback control laws. Our aim is to design such control law so as to minimize the stationary variance of the controlled output z, subject to the stationary SNR constraints imposed by the channel (see (3)). We can thus define the problem of interest as follows:

***Problem** 1.* Consider the NCS of Fig. 1, where G has the realization in (1), Assumption 1 holds, and the link between v and w is an SNR constrained additive white noise channel. Find

$$\left[\sigma_z^2\right]_\Gamma \triangleq \inf_{\substack{K\in\mathcal{S},\ 0\le P_{q_i}<\infty \\ P_{v_i}\le\Gamma_i P_{q_i}}} \sigma_z^2, \tag{6}$$

where $i \in \{1,\ldots,c\}$, $\Gamma \triangleq \{\Gamma_1,\ldots,\Gamma_c\}$, σ_z^2 is the stationary variance of z, and

$$\mathcal{S} \triangleq \left\{K\in\mathbb{R}^{m\times n} : \text{the closed loop of Fig. 1 is internally stable}\right\}. \qquad \square$$

In Problem 1, the notation $\left[\sigma_z^2\right]_\Gamma$ is used to emphasize the fact that σ_z^2 depends on the maximum admissible channel SNRs $\Gamma_1,\cdots,\Gamma_c$.

3. OPTIMAL DESIGN SUBJECT TO SNR CONSTRAINTS

In this section we provide a solution to Problem 1. To do so, we use results from the literature on control system design subject to upper bounds on the state covariance (see Skelton et al. (1997)).

When there is no feedback from x to u, a state space representation of the system of Fig. 1 is given by

$$x(k+1) = A_p x(k) + B_p u(k) + D_p\bar{d}(k),\ x(0)=x_0\,, \tag{7a}$$
$$z(k) = C_p x(k) + B_z u(k) + D_z\bar{d}(k), \tag{7b}$$

where

$$A_p \triangleq A + B_w\Delta C_v, \qquad B_p \triangleq B_u + B_w\Delta D_{uv},$$
$$D_p \triangleq \begin{bmatrix} D_{p_d} & D_{p_q}\end{bmatrix}, \qquad D_{p_d} \triangleq B_d + B_w\Delta D_{dv}\,,$$
$$D_{p_q} \triangleq B_w\Delta\,, \qquad C_p \triangleq C_z + D_{wz}\Delta C_v\,,$$
$$B_z \triangleq D_{uz} + D_{wz}\Delta D_{uv}\,, \qquad D_z \triangleq \begin{bmatrix} D_{z_d} & D_{z_q}\end{bmatrix},$$
$$D_{z_d} \triangleq D_{dz} + D_{wz}\Delta D_{dv}, \qquad D_{z_q} \triangleq D_{wz}\Delta,$$
$$\Delta \triangleq \left(I_{n\times n} - D_{wv}\right)^{-1}\,,$$

and

$$\bar{d}(k) \triangleq \operatorname{diag}\left\{d(k), q(k)\right\}\,.$$

The closed loop system that arises when the static state-feedback control law

$$u(k) = Kx(k), \quad K\in\mathbb{R}^{m\times n}, \tag{8}$$

is used to control the system described by (7) can be represented as

$$x(k+1) = A_{cl}x(k) + B_{cl}\bar{d}(k),\ \ x(0)=x_0\,, \tag{9a}$$
$$z(k) = C_{cl}x(k) + D_{cl}\bar{d}(k), \tag{9b}$$

where

$$A_{cl} \triangleq A_p + B_pK, \qquad B_{cl} \triangleq D_p, \tag{10a}$$
$$C_{cl} \triangleq C_p + B_yK, \qquad D_{cl} \triangleq D_y. \tag{10b}$$

***Lemma** 1.* Consider the discrete time LTI system in (9) with x_o and d satisfying Assumption 1a)-b), and q as in Definition 1. If a positive semidefinite matrix Λ is given, then the following statements are equivalent:

a) The system in (9) is asymptotically stable and the stationary covariance matrix of the output z is upper bounded by Λ, i.e.,

$$\lim_{k\to\infty}\mathcal{E}\left\{z(k)z(k)^T\right\} < \Lambda.$$

b) There exists $X>0$ such that

$$X > A_{cl}XA_{cl}^T + B_{cl}P_{\bar{d}}B_{cl}^T\,, \tag{11a}$$
$$\Lambda > C_{cl}XC_{cl}^T + D_{cl}P_{\bar{d}}D_{cl}^T\,, \tag{11b}$$

where $P_{\bar{d}} \triangleq \operatorname{diag}\{P_d, P_q\}$.

Proof. The result follows by using Lemma 6.1.2 in Skelton et al. (1997). ■

Lemma 1 allows one to characterizes all stabilizing LTI controllers that achieve a stationary output variance bounded from above by a given positive semidefinite matrix. With the help of this result, we are now in a position to state the main result of this section:

***Theorem** 1.* Consider Problem 1. Define the following optimization problem in the matrix variables Λ, X, Z and P_q (of appropriate dimensions):

$$\text{Find}: \quad \gamma \triangleq \inf\ \operatorname{trace}\{\Lambda\}\,, \tag{12}$$

$$\text{subject to:} \quad \begin{bmatrix} \Lambda & C_pX + B_zZ & D_{z_d}P_d & D_{z_q}P_q \\ \star & X & 0 & 0 \\ \star & \star & P_d & 0 \\ \star & \star & \star & P_q \end{bmatrix} > 0\,, \tag{13}$$

$$\begin{bmatrix} X & A_pX + B_pZ & D_{p_d}P_d & D_{p_q}P_q \\ \star & X & 0 & 0 \\ \star & \star & P_d & 0 \\ \star & \star & \star & P_q \end{bmatrix} > 0\,, \tag{14}$$

$$\begin{bmatrix} \Gamma_i\eta_iP_q\eta_i^T & C_{v_i}X + D_{uv_i}Z & D_{dv_i}P_d \\ \star & X & 0 \\ \star & \star & P_d \end{bmatrix} \ge 0\,, \tag{15}$$

where $i\in\{1,\ldots,c\}$, $\star$ correspond to entries that can be inferred by symmetry, and all the matrices involved are defined in (4), (5) and immediately after (7). Then:

(1) There exists a static state-feedback gain $K\in\mathcal{S}$ and noise variances P_{q_i}, $i\in\{1,\ldots,c\}$, satisfying $0\le P_{q_i}<\infty$ and $P_{v_i}\le\Gamma_iP_{q_i}$, if and only if the LMIs in (13)-(15) are feasible.
(2) If the optimization problem defined by (12)-(15) is feasible, then $\left[\sigma_z^2\right]_\Gamma = \gamma$. Moreover, if (Z_o, X_o, P_q^o) are the corresponding optimal values of (Z, X, P_q), then the choice of parameters $K = K_o \triangleq Z_oX_o^{-1}$ and $P_q = P_q^o$ guarantees that the closed loop system of Fig. 1 is internally stable, that the SNR constraints $P_{v_i}\le\Gamma_iP_{q_i}$ are satisfied $\forall i\in\{1,\cdots,c\}$, and that $\sigma_z^2=\gamma$.

Proof. By considering the definitions of A_{cl}, B_{cl}, C_{cl}, D_{cl} in (10), the inequalities (11) can be written in our case as

$$X > (A_p + B_p K)\, X\, (A_p + B_p K)^T + D_p P_{\bar{d}} D_p^T, \qquad (16)$$

$$\Lambda > (C_p + B_z K)\, X\, (C_p + B_z K)^T + D_z P_{\bar{d}} D_z^T. \qquad (17)$$

Using Schur complement (see, e.g., Boyd et al. (1994)), (16) can be written as

$$\begin{bmatrix} X & A_p + B_p K & D_p \\ \star & X^{-1} & 0 \\ \star & \star & P_{\bar{d}}^{-1} \end{bmatrix} > 0 \qquad (18)$$

Define $T \triangleq \operatorname{diag}\{I, X, P_{\bar{d}}\}$ where $P_{\bar{d}} \triangleq \operatorname{diag}\{P_d, P_q\}$. Then, (18) becomes equivalent to

$$T \begin{bmatrix} X & A_p + B_p K & D_p \\ \star & X^{-1} & 0 \\ \star & \star & P_{\bar{d}}^{-1} \end{bmatrix} T^T > 0$$

$$\iff \begin{bmatrix} X & A_p + B_p K X & D_p P_{\bar{d}} \\ \star & X & 0 \\ \star & \star & P_{\bar{d}} \end{bmatrix} > 0. \qquad (19)$$

Considering the definition of D_p in (7) we have that (19) is equivalent to

$$\begin{bmatrix} X & A_p X + B_p K X & D_{p_d} P_d & D_{p_q} P_q \\ \star & X & 0 & 0 \\ \star & \star & P_d & 0 \\ \star & \star & \star & P_q \end{bmatrix} > 0. \qquad (20)$$

Defining

$$Z \triangleq KX, \qquad (21)$$

and using this definition in (20), we get (14).

Similarly, (13) can be obtained from (17).

We can now state an optimization problem on the set of matrix variables X and Z where we aim to minimize the trace of Λ subject to (13) and (14), i.e.,

$$\begin{aligned} &\inf \operatorname{trace}\{\Lambda\}\ , \\ &\text{subject to} \quad (13), (14) \end{aligned} \qquad (22)$$

where the functional corresponds to an upper bound on the stationary variance of z (Lemma 1). However, this problem does not consider the SNR constraints imposed by the channel. These constraints (i.e., $P_{v_i} \le \Gamma_i P_{q_i}$, $\forall i \in \{1, \dots, c\}$) can be written as a function of the realization of the G (see (1)) and the controller K as

$$\begin{aligned} \Gamma_i \eta_i P_q \eta_i^T \ge \eta_i \left(C_v + D_{uv} K\right) X \left(C_v + D_{uv} K\right)^T \eta_i^T \\ + \eta_i D_{dv} P_d D_{dv}^T \eta_i^T, \end{aligned} \qquad (23)$$

where η_i has been defined in (5).

Using the procedure employed to obtain (13) and (14), one can rewrite (23) as in (15).

Adding the inequalities defined by (15) for every i to the optimization problem in (22), we complete the proof. ■

Theorem 1 provides a solution to Problem 1 in terms of the solution to a convex optimization problem subject to LMI constraints that, as such, can be solved by using standard numerical algorithms (Boyd et al. (1994); Grant and Boyd (2010)). A key feature of the problem at hand is that the decision variables include not only the static state-feedback controller K, but also a number of unknown covariance matrices P_{q_i}. Thus, the solution to Problem 1 presented in Theorem 1 required a (slight) modification of the standard procedures for writing optimal control problems in terms of LMIs.

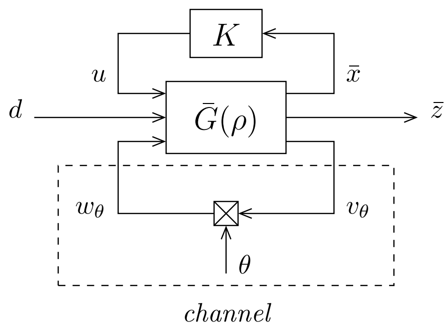

Fig. 2. NCS closed over a two-block erasure channel.

It is worth noting that the feasibility of (13)-(15) is not only sufficient, but also necessary for the existence of a stabilizing controller K and of noise covariances P_{q_i} satisfying the channel SNR constraints. Hence, one can use (13)-(15) to numerically characterize the set of all channel SNRs Γ_i that allow one to stabilize, by means of a static state-feedback controller, a given MIMO plant using multiple SNR constrained channels. (In the single scalar channel case, Braslavsky et al. (2007) and Silva et al. (2010) provide closed form characterizations for the minimal SNR compatible with stability for various networked architectures.)

4. AN APPLICATION: OPTIMAL CONTROL OVER UNRELIABLE CHANNELS

In this section, we use the results of Section 3 to solve an optimal control problem for NCSs closed over unreliable channels. In particular, we focus on the setup considered by Pulgar et al. (2010), which has been reproduced in Fig. 2. In that figure, all symbols with no bars are as defined before, $\bar{G}$ has the state space description

$$\begin{bmatrix} \bar{x}(k+1) \\ \hline v_\theta(k) \\ \bar{z}(k) \end{bmatrix} = \left[\begin{array}{c:ccc} \bar{A} & \bar{B}_u & \bar{B}_d & \bar{B}_w \\ \hdashline \bar{C}_v & \bar{D}_{uv} & 0 & 0 \\ \bar{C}_z & \bar{D}_{uz} & 0 & \bar{D}_{wz} \end{array}\right] \begin{bmatrix} \bar{x}(k) \\ \hline u(k) \\ d(k) \\ w_\theta(k) \end{bmatrix}, \qquad (24)$$

where $\bar{x}$ is the state, $\bar{x}(0) = \bar{x}_o$, $\bar{z}$ is an output, and the link between v_θ and w_θ is given by a two-block erasure channel:

Definition 2. The channel in Fig. 2, with input v_θ and output w_θ, is a two-block erasure channel if and only if, $\forall k \in \mathbb{N}_0, \forall v_\theta(k) \in \mathbb{R}^\ell$,

$$w_\theta(k) = \theta(k) v_\theta(k), \qquad (25)$$

where

$$\theta(k) \triangleq \operatorname{diag}\{\theta_1(k) I_{n_1}, \theta_2(k) I_{n_2}\}\ , \qquad (26)$$

$\theta_i(k) \in \{0, 1\}$, $\ell = n_1 + n_2$, θ_i is a sequence of i.i.d. Bernoulli random variables such that $\mathcal{P}\{\theta_i(k) = 1\} = p_i$, with $0 < p_i < 1$, θ_1 is independent of θ_2, and θ_i is independent of $(\bar{x}_0, d)$. □

Remark 2. In order to directly use the results in Pulgar et al. (2010), we assume that $\bar{G}$ is such that there exists no

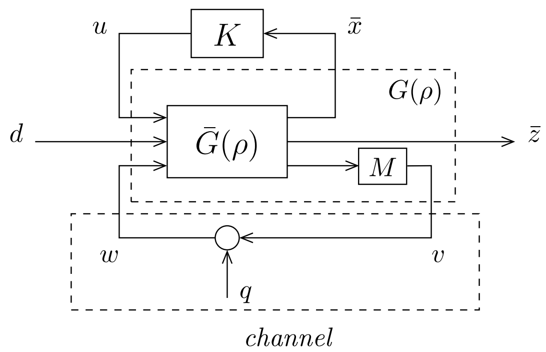

Fig. 3. Auxiliary NCS where the erasure channel has been replaced by a gain and an SNR constrained additive white noise channel.

feedthrough between (d, w_θ) and v_θ, and between d and $\bar{z}$. However, it is straightforward to extend the following results to more general cases. □

For the situation described above, we are interested in finding the state-feedback gain K that minimizes the stationary variance of the controlled output $\bar{z}$:

Problem 2. Consider the NCS in Fig. 2, where $\bar{G}$ has the realization in (24), $\bar{x}_o$ is a second order random variable, the disturbance d is as in Assumption 1b) and is uncorrelated with $\bar{x}_o$, and the link between v_θ and w_θ is a two-block erasure channel. Find

$$\left[\sigma_{\bar{z}}^2\right]_p \triangleq \inf_{K\in\mathcal{S}_M} \sigma_{\bar{z}}^2, \tag{27}$$

where $\sigma_{\bar{z}}^2$ is the stationary variance of $\bar{z}$, $p \triangleq \{p_1, p_2\}$, and

$$\mathcal{S}_M \triangleq \big\{K \in \mathbb{R}^{m\times n} : \text{the closed loop of Fig. 2 is stable in the mean square sense.}^2 \big\}.$$ □

By exploiting MJLSs theory (see do Val et al. (2002) and Costa et al. (2005)), Pulgar et al. (2010) provided a sufficient condition for the existence of $K \in \mathcal{S}_M$, and an upper bound on the best achievable performance $\left[\sigma_{\bar{z}}^2\right]_p$. If a solution to the problem of optimally designing mode-independent state-feedback controllers for MJLSs were available, then a characterization of $\left[\sigma_{\bar{z}}^2\right]_p$ could be readily obtained by proceeding as in Pulgar et al. (2010). However, to the best of our knowledge, that problem is still open in the MJLS literature.

We will now show how to use the results of Section 3 to solve Problem 2. To that end, we start by considering the auxiliary NCS of Figure 3, where the two-block erasure channel has been replaced by a matrix gain

$$M \triangleq \operatorname{diag}\{p_1 I_{n_1}, p_2 I_{n_2}\}, \tag{28}$$

and an SNR constrained additive noise channel with $c = 2$ and

$$\Gamma_1 = \frac{p_1}{1-p_1}, \quad \Gamma_2 = \frac{p_2}{1-p_2}. \tag{29}$$

Note that the LTI system inside the uppermost dashed box in Figure 3 plays the role of G in Figure 1.

For the sake of clarity, we will henceforth use $\bar{z}^M$ and $\bar{z}^L$ to refer to the signal $\bar{z}$ in the switched system of Figure 2, and in the LTI system of Figure 3, respectively.

[2] See, e.g., Costa et al. (2005).

Theorem 2. Consider Problem 2. Also consider, under the same assumptions as those of Problem 2, the NCS of Figure 3, where M is as in (28) and the link between v and w is an SNR constrained additive noise channel with $c = 2$ and SNRs given by (29). Then,

$$\left[\sigma_{\bar{z}^M}^2\right]_p = \inf_{\substack{K\in\mathcal{S},\ 0\le P_{q_i}<\infty \\ P_{v_i}=\Gamma_i P_{q_i}}} \sigma_{\bar{z}^L}^2. \tag{30}$$

where $\mathcal{S}$ is the set of all static gains K that make the LTI system of Fig. 3 internally stable.

Moreover, the static state-feedback controller $K \in \mathcal{S}$ that solves the right-hand side optimization problem in (30) is also the controller $K \in \mathcal{S}_M$ that solves Problem 2.

Proof. Immediate from Theorem 2 and Corollaries 2 and 3 in Pulgar et al. (2010). ■

Theorem 2 states that solving Problem 2 is, essentially, equivalent to solving Problem 1 for a specific choice for the LTI system G, provided that the SNR constraints of Problem 1 are active at the optimum. By exploiting Theorem 2, the following characterization of the solution to Problem 2 becomes immediate:

Corollary 1. Consider Problem 2. Define the following optimization problem in the matrix variables Λ, X, Z and P_q (of appropriate dimensions):

$$\text{Find}: \quad \gamma \triangleq \inf \operatorname{trace}\{\Lambda\} \tag{31}$$

$$\text{subject to:} \quad \begin{bmatrix} \Lambda & \bar{C}_p X + \bar{B}_z Z & \bar{D}_{wz} P_q \\ \star & X & 0 \\ \star & \star & P_q \end{bmatrix} > 0, \tag{32}$$

$$\begin{bmatrix} X & \bar{A}_p X + \bar{B}_p Z & \bar{B}_d P_d & \bar{B}_w P_q \\ \star & X & 0 & 0 \\ \star & \star & P_d & 0 \\ \star & \star & \star & P_q \end{bmatrix} > 0, \tag{33}$$

$$\begin{bmatrix} \Gamma_1 \eta_1 P_q \eta_1^T & \bar{C}_{v_1} X + \bar{D}_{uv_1} Z \\ \star & X \end{bmatrix} \ge 0, \tag{34}$$

$$\begin{bmatrix} \Gamma_2 \eta_2 P_q \eta_2^T & \bar{C}_{v_2} X + \bar{D}_{uv_2} Z \\ \star & X \end{bmatrix} \ge 0, \tag{35}$$

where $\star$ correspond to entries that can be inferred by symmetry, and the remaining matrices are defined in terms of the state space description of $\bar{G}$ (see (24)) as follows:

$$\bar{A}_p \triangleq \bar{A} + \bar{B}_w \bar{C}_v, \qquad \bar{B}_p \triangleq \bar{B}_u + \bar{B}_w \bar{D}_{uv},$$
$$\bar{C}_p \triangleq \bar{C}_z + \bar{D}_{wz} \bar{C}_v, \qquad \bar{B}_z \triangleq \bar{D}_{uz} + \bar{D}_{wz} \bar{D}_{uv},$$
$$\bar{C}_{v_i} \triangleq \eta_i \bar{C}_v, \qquad \bar{D}_{uv_i} \triangleq \eta_i \bar{D}_{uv},$$

where, η_i is as in (5).

Then:

(1) There exists a static state-feedback gain $K \in \mathcal{S}_M$ if and only if the LMIs in (32)-(35) are feasible, and the optimal values of (X, Z, P_q), say (X_o, Z_o, P_q^o), are such that, $\forall i \in \{1, 2\}$,

$$\Gamma_i \eta_i P_q^o \eta_i^T = (\bar{C}_{v_i} X_o + \bar{D}_{uv_i} Z_o) X_o^{-1} (\bar{C}_{v_i} X_o + \bar{D}_{uv_i} Z_o)^T. \tag{36}$$

(2) If the optimization problem defined by (31)-(35) is feasible, and the optimal values of (X, Z, P_q) are such that (36) holds for every $i \in \{1, 2\}$, then $\left[\sigma_{\bar{z}^M}^2\right]_p = \gamma$.

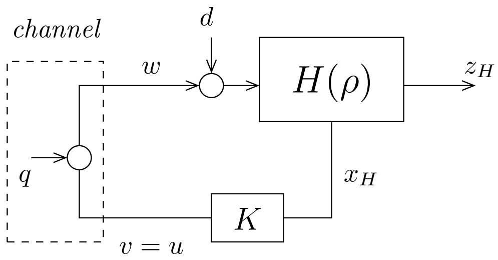

Fig. 4. Networked system considered in the example of Section 5.

Moreover, the choice $K = K_o \triangleq Z_o X_o^{-1}$ guarantees that the NCS of Fig. 2 is mean square stable and that $\sigma^2_{\bar{z}^M} = \gamma$.

Proof. The result follows immediately upon using Theorem 1, its proof, and Theorem 2. ■

As opposed to the results by Pulgar et al. (2010) (based upon do Val et al. (2002)), Corollary 1 provides a solution to Problem 2. In particular, Corollary 1 establishes both necessary and sufficient conditions for the existence of $K \in \mathcal{S}_M$, and an exact characterization of the best achievable performance in terms of the solution to a convex optimization problem. It is worth noting that our result solves the optimal mode-independent static state-feedback control problem for a specific class of MJLSs. [3] Whether or not our result can be extended to more general MJLSs is a subject for future research.

We end this section with a remark on computational complexity. The approach based on MJLS theory adopted by Pulgar et al. (2010) provides an *upper bound on the solution to* Problem 2 in terms of 2^{c+1} LMIs. If it was possible to show that Theorem 2 holds for an arbitrary number of channels (not just $c = 2$ channels), then the approach in the present paper would yield an *exact characterization of the solution to* Problem 2 in terms of only $2 + c$ LMIs. This might imply an important saving in terms of computational burden, when c is large.

5. SIMULATION STUDY

In this section, we present an example to illustrate the results of this paper. Consider the control system of Fig. 4, where the plant H has the state space description

$$\begin{bmatrix} x_H(k+1) \\ z_H(k) \end{bmatrix} = \begin{bmatrix} A_H & B_H & B_H \\ C_H & 0 & 0 \end{bmatrix} \begin{bmatrix} x_H(k) \\ w(k) \\ d(k) \end{bmatrix},$$

with

$$A_H = \begin{bmatrix} 1.1 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix},$$

$$B_H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ \sqrt{2} & 0 \end{bmatrix}, \quad C_H = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \sqrt{2} \end{bmatrix},$$

and where x_H is the state, z_H the output, d is a zero mean white noise disturbance with covariance matrix $P_d = I_2$, and w is the control input.

In Fig. 4, the controller output $u \triangleq [u_1 \quad u_2]^T$ has to be transmitted over an SNR constrained additive noise channel with $n_1 = n_2 = 1$, and maximum admissible SNRs

$$\Gamma_1 = 100 \quad \text{and} \quad \Gamma_2 = 1.5\,.$$

The control objective is to minimize the stationary variance of z_H.

The convex problem proposed by Pulgar et al. (2010) to determine an upper bound on the best achievable performance turns out to be unfeasible. [4] However, the results of Section 3 allow one to actually find the best achievable performance, namely $[\sigma_z^2]_{\Gamma_1,\Gamma_2} = [\sigma_z^2]_{100,1.5} = 34.1587$. The corresponding optimal controller gain is given by

$$K_o = \begin{bmatrix} -1.0087 & 0.5941 & -0.2231 & -0.0752 \\ 0.0783 & -1.8081 & 0.0041 & -0.0051 \end{bmatrix}, \qquad (37)$$

and the optimal noise variances by

$$P^o_{q_1} = 0.0109, \quad P^o_{q_2} = 5.4635. \qquad (38)$$

The system of Fig. 4 was simulated [5] using the parameters in (37) and (38), obtaining a *measured* stationary variance for z_H equal to 34.1257, and *measured* channel SNRs given by

$$\frac{P_{v_1}}{P_{q_1}} = 1.4990 \quad \text{and} \quad \frac{P_{v_1}}{P_{q_1}} = 99.8597\,.$$

As expected, the simulation results match our theoretical predictions.

Consider now the case where the channel is such that the maximum allowable SNRs are inverted, i.e., assume that

$$\Gamma_1 = 1.5, \quad \Gamma_2 = 100.$$

The results in Pulgar et al. (2010) yield the upper bound $[\sigma_z^2]_{\Gamma_1,\Gamma_2} = [\sigma_z^2]_{1.5,100} \leq 115.244$. In turn, the approach proposed in this paper allows one to see that

$$[\sigma_z^2]_{1.5,100} = 10.9503\,.$$

and that the associated optimal static state-feedback gain is given by

$$K_o = \begin{bmatrix} -0.7826 & 0.5497 & -0.1477 & -0.0294 \\ 0.7109 & -3.2082 & 0.0642 & -0.0845 \end{bmatrix},$$

and that the optimal noise variances satisfy

$$P^o_{q_1} = 0.2702, \quad P^o_{q_2} = 0.0267.$$

We simulated the resulting networked system and, again, the simulation results matched the theoretical ones quite well.

We conclude from the above that the system performance is better in the second case. Thus, u_2 should be transmitted over the more reliable channel, whilst u_1 should be transmitted over the less reliable one.

It is also worth noting that, although the results by Pulgar et al. (2010) allowed one to obtain an upper bound on the best achievable performance in the second case, this upper bound is over-conservative (about ten times the actual optimal performance).

[3] Namely, those MJLSs that can be written as in Figure 2.

[4] We used CVX for Matlab (Grant and Boyd (2010)).

[5] All simulation results are averages over 100 simulations, each 10^4 samples long.

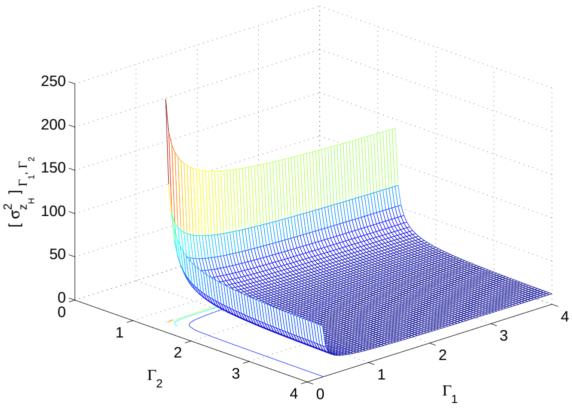

Fig. 5. Stationary variance of z_H, $\sigma^2_{z_H}$, as a function of Γ_1 and Γ_2.

We end this section by showing, in Fig. 5, a plot of the best achievable performance as a function of the channel SNRs Γ_1 and Γ_2.

6. CONCLUSIONS

This paper has studied the problem of optimal control system design for MIMO LTI systems closed over multiple SNR constrained channels. For this type of systems, and by focusing on the state-feedback case, we provided an LMI based convex optimization problem to fully characterize the best achievable performance. Our methodology was also applied to optimal control system design for MIMO LTI systems closed over unreliable channels.

Future work should focus on dynamic output-feedback control laws, and on an extension of Theorem 2 to the n-channel case.

REFERENCES

Antsaklis, P. and Baillieul, J. (2007). Special issue on technology of networked control systems. *Proceedings of the IEEE*, 95(1), 5–8.

Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V. (1994). *Linear Matrix Inequalities in System and Control Theory*. SIAM.

Braslavsky, J., Middleton, R., and Freudenberg, J. (2007). Feedback stabilization over signal-to-noise ratio constrained channels. *IEEE Transactions on Automatic Control*, 52(8), 1391–1403.

Costa, O., Fragoso, M., and Marques, R. (2005). *Discrete Time Markov Jump Linear Systems*. Springer.

Cover, T. and Thomas, J. (2006). *Elements of Information Theory*. John Wiley and Sons, Inc., 2nd edition.

do Val, J., Geromel, J., and Gonalves, A. (2002). The H_2-control for jump linear systems:cluster observations of the markov state. *Automatica*, 38, 343–349.

Freudenberg, J.S., Middlenton, R.H., and Solo, V. (2010). Stabilization and disturbance attenuation over a gaussian communication channel. *IEEE Transactions on Automatic Control*, AC–55(3), 795–799.

Freudenberg, J., Middleton, R., and Braslavsky, J. (2007). Stabilization with disturbance attenuation over a Gaussian channel. In *Proceedings of the 46th IEEE Conference on Decision and Control*. New Orleans, USA.

Geromel, J., Gonalves, A., and Fioravanti, A. (2009). Dynamic output feedback control of discrete-time markov jump linear systems through linear matrix inequalities. *SIAM*, 48(2), 573–593.

Grant, M. and Boyd, S. (2010). CVX: Matlab software for disciplined convex programming, version 1.21. `http://cvxr.com/cvx`.

Nair, G., Fagnani, F., Zampieri, S., and Evans, R. (2007). Feedback control under data rate constraints: An overview. *Proceedings of the IEEE*, 95(1), 108–137.

Pulgar, S., Silva, E., and Salgado, M. (2010). Upper bounds on best achievable performance subjecto to snr constraints in two-channel networked architectures. In *To be presented at 49th IEEE Conference on Decision and Control*. Atlanta, Georgia, USA.

Rojas, A., Braslavsky, J., and Middleton, R. (2008). Channel signal-to-noise ratio constrained feedback control: performance and robustness. *IET Control Theory & Applications*, 2(7), 595–605.

Schenato, L., Sinopoli, B., Franceschetti, M., Poolla, K., and Sastry, S. (2007). Foundations of control and estimation over lossy networks. *Proceedings of the IEEE*, 95(1), 163 – 187.

Shu, Z., Lam, J., and Xiong, J. (2010). Static output-feedback stabilization of dicrete-time markovian jump linear systems: A system augmentation approach. *Automatica*, 46, 687–694.

Silva, E., Goodwin, G., and Quevedo, D. (2010). Control system design subject to SNR constraints. *Automatica*, 46(2), 428–436.

Skelton, R., Iwasaki, T., and Grigoriadis, K. (1997). *A Unified Algebraic Approach to Linear Control Design*. Taylor& Francis.

Xiong, J. and Lam, J. (2007). Stabilization of linear systems over networks with bounded packet loss. *Automatica*, 43, 80–87.

Zhang, L., Shi, Y., Chen, T., and Huang, B. (2005). A new method for stabilization of networked control systems with random delays. *IEEE Transactions on Automatic Control*, 50(8), 1177–1181.

Zhou, K., Doyle, J., and Glover, K. (1996). *Robust and optimal control*. Prentice Hall.