Control System Design over Unreliable Channels [⋆]

Eduardo I. Silva [a], Sebastián A. Pulgar [a]

[a] *Departamento de Electrónica, Universidad Técnica Federico Santa María, Valparaíso, Chile*

Abstract

This paper studies LTI control systems comprising a scalar unreliable channel in the feedback path. We show that, when such a channel drops data according to a Bernoulli process, it becomes equivalent, as far as second order moments are concerned, to an additive white noise channel with an instantaneous signal-to-noise ratio constraint. This equivalence allows one to characterize the set of all LTI controllers that achieve mean square stability in a general control architecture closed over a scalar unreliable channel, and also enables one to optimally design LTI controllers using standard tools. To illustrate our results, we study the (LTI dynamic output feedback) control of SISO plants subject to data dropouts. For this situation, we establish closed form necessary and sufficient conditions on the minimal successful transmission probability that allows one to design LTI controllers that achieve MSS, when either TCP or UDP-like communication protocols are employed.

Key words: Networked control systems; data-loss; unreliable channels; signal-to-noise ratio.

1 Introduction

Networked control systems (NCSs) are control systems subject to communication constraints [1]. This paper focuses on NCSs closed over channels prone to data-loss.

The simplest model for data-loss assumes that uncorrupted data is received at the receiving end with a probability that is constant over time [6, 7, 11, 19, 20]. This model for data-loss has received the most attention in the literature, and is also the model adopted in this paper. Other models have been considered in, e.g, [10, 15, 27].

Early work in the area of control subject to data-loss includes [9,17]. That work studied simple dropout compensation schemes, where previously received samples are held when a dropout occurs, or where lost data is replaced by zeros. The actual design of dropout compensators has been addressed in [16]. A key conclusion in [16] is that, for the architecture considered there, the resulting switched system is equivalent, in steady state, to a linear system with an external noise source having a variance proportional to the variance of a signal within the loop. This result is then used to design a dropout compensator so as to minimize the stationary plant output variance.

The performance gains arising from data-dropout compensation may be marginal in some situations. If that is the case, then a complete controller re-design is needed. For example, [13, 20] use Markov jump linear system (MJLS) theory [5] to synthesize controllers that minimize an $\mathcal{H}_\infty$ functional in the presence of i.i.d. data-dropouts. Another related work is [7], where the control of SISO plants using LTI controllers and a class of LTI-filter-based switching compensators is considered. For such architectures, [7] shows that it is possible to view the resulting NCS as an uncertain linear feedback system where the uncertainty block accounts for the data-dropouts. Using such a reformulation, and standard robust control tools, [7] studies mean square stabilization in different setups. The framework in [7] is further extended in [6]. In the latter work, a general MIMO control problem is studied, where

⋆ This paper was not presented at any IFAC meeting. Corresponding author: E.I. Silva, Departamento de Electrónica, Universidad Técnica Federico Santa María, Casilla 100-V, Valparaíso, Chile. Tel. +56-32-2654495

Email address: `eduardo.silva@usm.cl` (Eduardo I. Silva).

communication takes place over a fading network. Section 8 in [6] particularizes the results to a case that employs a single unreliable channel that drops data in an i.i.d. fashion. In this case, and assuming a single input plant whose state is noiselessly measured, it is shown in [6] that an LTI controller that achieves mean square stability (MSS) can be found, if and only if the probability of successful transmission is greater than a function of the product of the unstable plant poles (see also [7]). No performance related questions are considered in [6, 7].

Control subject to data-dropouts has also been addressed from the perspective of classical LQG control theory in, e.g., [12, 19]. The work [12] was one of the first to point out that there exist control problems where fundamental differences arise depending on whether or not there exist acknowledgements that testify successful transmissions. If such acknowledgements are available (TCP-like protocols), then the classical separation principle holds, whilst no separation holds when no acknowledgements are available (UDP-like protocols) [19]. It has also been shown in [12,19] that, consistent with the results in [6, 7], there exists a region in the plane of successful transmission probabilities within which MSS can be guaranteed. As expected, these regions are protocol dependant. Another conclusion in [19] is that LTI control policies are suboptimal when dealing with data-dropouts. Indeed, in the UDP-like protocol case, optimal control laws are non-linear in general and they do not seem to admit a simple characterization [19]. This motivated the study of optimal linear controllers pursued in [24]. The works [12, 19, 24] assume that lost data in the controller to actuator link is replaced by zeros. A more general approach is considered in [8], where the architecture is enriched with an estimator that accounts for lost data at the actuator side. By doing so, it is shown in [8] that the channel reliability required to achieve stability is, in general, smaller than that identified in [6, 7, 12, 19]. It is also shown in [8] that the architecture proposed there is optimal *irrespective* of the actual channel dropout profile.

In this paper we consider a general LTI control system comprising a scalar unreliable channel in the feedback path, and assume that data-loss is governed by a Bernoulli process. As a first contribution, we show that such an architecture is equivalent, as far as second order moments are concerned, to an LTI control architecture where the unreliable channel has been replaced by an additive white noise channel with an instantaneous signal-to-noise ratio constraint. This key result allows one to obtain a closed form characterization of the set of all LTI controllers that achieve mean square stability in the considered architecture. It also allows one to address performance related questions. In particular, we show that optimal LTI controllers can be characterized in terms of a standard quadratic control problem, coupled with a line search. These results are then used to study the LTI dynamic output feedback control of SISO plants subject to data dropouts in the feedback path. For this situation, and as a second contribution, we establish both necessary and sufficient conditions on the minimal successful transmission probability that allows one to design LTI controllers that achieve MSS, when either TCP or UDP-like communication protocols are employed. Our conditions are given in closed form and show, when no acknowledgements are available, that the plant non-minimum phase zeros and relative degree influence the channel reliability required to achieve MSS.

The results in this paper go beyond the results in [16], where no explicit study of the interplay between MSS and dropout probability is performed. Our results can be seen as an alternative to the robust-control related framework presented in [6, 7] for the design of LTI control architectures over unreliable channels. But, unlike [6, 7], we study performance related questions which, by virtue of our results, can be addressed in a simple manner. When applied to the control of SISO plants, our results allow one to establish necessary and sufficient closed form conditions for the existence of LTI controllers that achieve MSS in an output-feedback UDP-like protocol based control architecture, whereas the available results in [12, 19] give only sufficient or necessary conditions depending on the plant structure.

The remainder of the paper is organized as follows: Section 2 presents the notation used throughout the paper. Section 3 presents assumptions and states the problem of interest. Section 4 shows that the NCS architecture under consideration is equivalent (in the sense indicated above) to a standard LTI feedback loop with an instantaneous signal-to-noise ratio constraint. Section 5 uses the results of Section 4 to rewrite the problem of optimal LTI controller design over unreliable channels, in a form that is amenable to solution using standard tools. Section 6 focuses on SISO plants and presents a numerical example. Section 7 draws conclusions.

2 Notation

$\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}$ refer to the real, complex and natural numbers, respectively; $\mathbb{N}_0 \triangleq \{0, 1, \cdots\}$ and $\mathbb{R}_0^+ \triangleq \{x \in \mathbb{R} : 0 \leq x < \infty\}$. $\mathcal{P}\{\cdot\}$ stands for the probability of $(\cdot)$ and $\mathcal{E}\{\cdot\}$ denotes the expectation operator. Given any scalar $x \in \mathbb{C}$, $|x|$ denotes its magnitude. Given any matrix M, M^T and M^H denote transpose and conjugate transpose, respectively. We use z as both the argument of the z-transform and as the forward shift operator, where the meaning is clear from the context.

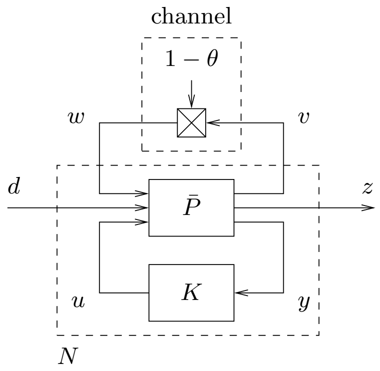

Fig. 1. LTI control system closed over a scalar erasure channel.

The set of all proper real rational discrete-time transfer functions is denoted by $\mathcal{R}_p$, and the subset of $\mathcal{R}_p$ containing all stable and proper transfer functions is denoted by $\mathcal{RH}_\infty$. Any real rational discrete time transfer function $A(z)$ with no poles on the unit circle belongs to $\mathcal{L}_2$. For each $A(z) \in \mathcal{L}_2$ we define the 2−norm as usual, and denote it by $||\cdot||_2$ [26]. We usually omit the z argument from transfer functions and write A instead of $A(z)$.

Random processes are always assumed to be real and defined for $k \in \mathbb{N}_0$. We abbreviate $\{x(k)\}_{k\in\mathbb{N}_0}$ by x. For any process x (analogous notation is used with random variables), we define $\mu_x(k) \triangleq \mathcal{E}\{x(k)\}$, $R_x(k+\tau,k) \triangleq \mathcal{E}\left\{(x(k+\tau)-\mu_x(k+\tau))(x(k)-\mu_x(k))^T\right\}$, $P_x(k) \triangleq R_x(k,k)$, $\sigma_x^2(k) \triangleq \text{trace}\{P_x(k)\}$. We refer to $P_x(k)$ as the covariance of x, and to $\sigma_x^2(k)$ as the variance of x. Provided they exist, we also define the stationary covariance and the stationary variance of x via $P_x \triangleq \lim_{k\to\infty} P_x(k)$ and $\sigma_x^2 \triangleq \lim_{k\to\infty}\sigma_x^2(k)$, respectively. If x is a wide sense stationary (wss) (asymptotically wss) process, then $S_x(e^{j\omega})$ denotes its (stationary) power spectral density (PSD) and $\Omega_x(z)$ denotes any spectral factor of $S_x(e^{j\omega})$, i.e., $S_x(e^{j\omega}) \triangleq \Omega_x(e^{j\omega})\Omega_x(e^{j\omega})^H$. We say that a random variable (process) is a second order one if and only if it has finite mean and finite second order moments (for all time instants $k \in \mathbb{N}_0$ and also when $k \to \infty$). We usually omit the $e^{j\omega}$ argument in PSDs and spectral factors.

3 Problem Definition

This paper focuses on the NCS architecture of Fig. 1. In that figure, $\bar{P}$ is a generalized LTI plant,[1] K is an LTI controller to be designed, d models disturbances, y is a measurable plant output, u is the control input, z is an output related to closed loop performance, and the channel between v and w is a *scalar analog erasure channel.* That is, the channel input v and output w take values in $\mathbb{R}$ and are related via

$$w(k) \triangleq (1-\theta(k))v(k), \quad \forall k \in \mathbb{N}_0, \tag{1}$$

where θ is a process that models data dropouts, and $\theta(k) \in \{0,1\}$ $\forall k \in \mathbb{N}_0$.

We denote the state of $\bar{P}$ by $x_{\bar{P}}$ and that of K by x_K. We define $x \triangleq [x_{\bar{P}}^T \;\; x_K^T]^T$ and denote the joint initial state by $x_o \triangleq [x_{\bar{P}}(0)^T \;\; x_K(0)^T]^T$.

Assumption 1

(a) x_o is a second order random variable with mean μ_o and covariance P_o. The disturbance d is a zero mean second order white noise sequence, uncorrelated with x_o, and having covariance $P_d \triangleq \Omega_d\Omega_d^H$.

(b) θ is a sequence of i.i.d. random variables such that $\theta(k) \in \{0,1\}$ and $\mathcal{P}\{\theta(k)=0\} = p$, $\forall k \in \mathbb{N}_0$. Moreover, θ is independent of (x_o, d). ■

The assumptions regarding initial states and disturbance signals are standard. Our second assumption implies that the channel under consideration drops data in an i.i.d. fashion (as in [6, 7, 11, 19, 20]).

In order to guarantee the well-posedness of the feedback loop of Fig. 1 for any choice of controller K and any successful transmission probability p, we introduce the following additional assumption:

[1] Unless explicitly stated otherwise, all signals and systems in this paper are of arbitrary dimensions.

Assumption 2 $\bar{P}$ *is such that:*

(a) If the channel is absent (i.e., no feedback from v to w is present) and K is proper and LTI, then the system of Fig. 1 is well posed and the transfer function from w to v is strictly proper.

(b) If K is absent and $w = v$, then the system of Fig. 1 is well posed and the transfer function from u to y is strictly proper. ■

Denote by N the LTI system resulting from the interconnection of $\bar{P}$ and K in the absence of the channel (see dashed box in Fig. 1). By virtue of Assumption 2(a), a suitable state space description of N is

$$\begin{bmatrix} x(k+1) \\ z(k) \\ v(k) \end{bmatrix} = \begin{bmatrix} A & B_d & B_w \\ C_z & D_{dz} & D_{wz} \\ C_v & D_{dv} & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \\ w(k) \end{bmatrix}, \quad k \in \mathbb{N}_0, \quad x(0) = x_o, \tag{2}$$

where all matrices are functions of the matrices in the state space descriptions of $\bar{P}$ and K, and $x(k) \in \mathbb{R}^{n_x}$ $\forall k \in \mathbb{N}_0$. The switching system of Fig. 1 can thus be described by

$$x(k+1) = A(\theta(k))x(k) + B(\theta(k))d(k), \quad k \in \mathbb{N}_0, \quad x(0) = x_o, \quad \theta(0) = \theta_o \tag{3a}$$
$$z(k) = C(\theta(k))x(k) + D(\theta(k))d(k), \tag{3b}$$

where $\theta_o \in \{0, 1\}$ is the random variable corresponding to the initial channel state, and

$$A(\theta(k)) \triangleq A + (1 - \theta(k))B_w C_v, \qquad B(\theta(k)) \triangleq B_d + (1 - \theta(k))B_w D_{dv}, \tag{4a}$$
$$C(\theta(k)) \triangleq C_z + (1 - \theta(k))D_{wz} C_z, \quad D(\theta(k)) \triangleq D_{dz} + (1 - \theta(k))D_{wz} D_{dv}. \tag{4b}$$

The state space description in (3) corresponds to that of a standard Markov jump linear system (MJLS) [5]. We adopt the following notion of stability throughout this paper (see also [5, 6, 14]):

Definition 3 *The system described by* (3)*, where* (x_o, θ, d) *satisfies Assumption 1, is mean square stable (MSS [2]) if and only if there exist finite* $\mu \in \mathbb{R}^{n_x}$ *and finite* $M \in \mathbb{R}^{n_x \times n_x}$*,* $M \geq 0$*, such that*

$$\lim_{k\to\infty} \mathcal{E}\{x(k)\} = \mu, \quad \lim_{k\to\infty} \mathcal{E}\left\{x(k)x(k)^T\right\} = M,$$

regardless of (x_o, θ_o). ■

For the MJLS in (3), a characterization of MSS is readily available:

Theorem 4 *Consider the MJLS system in* (3)*, suppose that Assumption 1 holds, and define* $A_p \triangleq A + pB_w C_v$*. The following statements are equivalent:*

(1) The system in (3) *is MSS.*

(2) There exists $P > 0$ *such that* $A_p P A_p^T + p(1-p)B_w C_v P C_v^T B_w^T < P$.

(3) The state x is a second order asymptotically wss process, i.e., there exist $\mu_x \in \mathbb{R}^{n_x}$ *and* $\{Q_x(\tau) \in \mathbb{R}^{n_x \times n_x}, \tau \in \mathbb{N}_0\}$*, both not depending on* (x_o, θ_o)*, such that* $\lim_{k\to\infty} \mathcal{E}\{x(k)\} = \mu_x$ *and* $\lim_{k\to\infty} \mathcal{E}\left\{x(k+\tau)x(k)^T\right\} = Q_x(\tau)$.

PROOF. According to Corollary 3.26 in [5], the system in (3) is MSS if and only if there exists $P > 0$ such that

$$pA(0)PA(0)^T + (1-p)A(1)PA(1)^T < P. \tag{5}$$

Using Fact 18 in the Appendix, and the definitions of $A(\theta(k))$ and A_p, we conclude that (5) is equivalent to the inequality in *(2)* and thus *(1)* $\Leftrightarrow$ *(2)*. The equivalence between Parts *(1)* and *(3)* follows from Theorem 3.33 in [5]. □

The following corollary of Theorem 4 will be useful in the sequel:

[2] We will use MSS for both “mean square stable” and “mean square stability”.

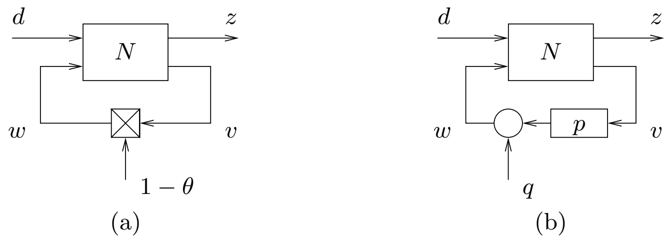

Fig. 2. System N with feedback over (a) erasure channel with dropout probability $1-p$, and (b) additive noise channel with gain p.

Corollary 5 *Consider the MJLS system in* (3) *and suppose that Assumption 1 holds. If the system* (3) *is MSS, then z is a second order asymptotically wss process.*

PROOF. Immediate from Part *(3)* of Theorem 4, the definition of z, and Assumption 1. $\square$

It follows from Corollary 5 that, provided the NCS under consideration is MSS, all signals within the loop become asymptotically wss processes and, as such, they have well-defined stationary PSDs, covariances, etc.

We can now state the problem of interest in this paper:

Problem 6 *Consider the NCS of Fig. 1 and suppose that Assumptions 1 and 2 hold. For a given successful transmission probability $p \in (0,1)$, find (or prove the problem infeasible)*

$$\left[\sigma_z^2\right]_p \triangleq \inf_{K \in \mathcal{S}_p} \sigma_z^2, \tag{6}$$

where σ_z^2 is the stationary variance of z, and $\mathcal{S}_p \triangleq \{K \in \mathcal{R}_p : \text{the loop in Fig. 1 is MSS}\}$. $\blacksquare$

To address Problem 6, we start by showing in Section 4 that the NCS of Fig. 1 can be analyzed by replacing the unreliable channel by an additive noise channel with an instantaneous signal-to-noise ratio (SNR) constraint.

4 Data Dropouts as instantaneous SNR Constraints

We redraw Fig. 1 as shown in Fig. 2(a), and consider an auxiliary situation where the scalar erasure channel of Fig. 2(a) has been replaced by a scalar additive noise channel plus a gain equal to the successful transmission probability p (see Fig. 2(b)). In Fig. 2(b), q is a noise source satisfying the following:

Assumption 7 *The signal q is a scalar zero mean white noise sequence, uncorrelated with (x_o, d), and having a covariance $P_q(k)$ that satisfies the instantaneous SNR constraint*

$$P_q(k) = p(1-p)P_v(k), \quad \forall k \in \mathbb{N}_0, \tag{7}$$

where $P_v(k)$ corresponds to the covariance of v in the LTI systems of Fig. 2(b). $\blacksquare$

Given (2), the LTI system of Fig. 2(b) can be described by

$$x(k+1) = A_p x(k) + B_p d(k) + B_w q(k), \quad k \in \mathbb{N}_0, \quad x(0) = x_o \tag{8a}$$
$$z(k) = C_p x(k) + D_p d(k) + D_{wz} q(k), \tag{8b}$$

where

$$A_p \triangleq A + pB_wC_v, \qquad B_p \triangleq B_d + pB_wD_{dv}, \tag{9}$$
$$C_p \triangleq C_z + pD_{wz}C_z, \quad D_p \triangleq D_{dz} + pD_{wz}D_{dv}. \tag{10}$$

Also, we have that

$$v(k) = C_v x(k) + D_{dv} d(k) \tag{11}$$

holds in both the system of Fig. 2(a) and that of Fig. 2(b).

In the sequel, we will sometimes use M and L superscripts to refer to quantities related to the MJLS of Fig. 2(a), or to the LTI system of Fig. 2(b), respectively. (E.g., $\mu_x^M(k)$ refers to the mean of $x(k)$ in Fig. 2(a), etc.)

Lemma 8 *Consider the MJLS of Fig. 2(a) and the LTI system of Fig. 2(b), where N has the realization given in (2). If Assumptions 1 and 7 hold, then $\forall k, \tau \in \mathbb{N}_0$:*

(1) $\mu_x^L(k) = \mu_x^M(k)$ and $R_x^L(k+\tau, k) = R_x^M(k+\tau, k)$.
(2) $\mu_z^L(k) = \mu_z^M(k)$ and $R_z^L(k+\tau, k) = R_z^M(k+\tau, k)$.

PROOF.

(1) The first claim is immediate. To prove the second claim, assume, without loss of generality, that $\mu_o = 0$. Use of (3a) and Assumption 1 yields

$$\begin{aligned} P_x^M(k+1) &= pA(0)P_x^M(k)A(0)^T + (1-p)A(1)P_x^M(k)A(1)^T + pB(0)P_dB(0)^T + (1-p)B(1)P_dB(1)^T \\ &= \Psi_A(P_x^M(k)) + \Psi_B(P_x^M(k)), \end{aligned} \tag{12}$$

where we have used Fact 18 in the Appendix, and have defined, for any suitable matrix P,

$$\Psi_A(P) \triangleq A_pPA_p^T + p(1-p)B_wC_vPC_v^TB_w^T, \quad \Psi_B(P) \triangleq B_pPB_p^T + p(1-p)B_wD_{dv}P_dD_{dv}^TB_w^T.$$

On the other hand, (8a) implies

$$\begin{aligned} P_x^L(k+1) &= A_pP_x^L(k)A_p^T + B_pP_dB_p^T + B_wP_q(k)B_w^T \\ &\stackrel{(a)}{=} \Psi_A(P_x^L(k)) + \Psi_B(P_x^L(k)) - p(1-p)B_w\left(C_vP_x^L(k)C_v^T + D_{dv}P_dD_{dv}^T\right)B_w^T + B_wP_q(k)B_w^T \\ &\stackrel{(b)}{=} \Psi_A(P_x^L(k)) + \Psi_B(P_x^L(k)) - B_w\left(p(1-p)P_v^L(k) - P_q(k)\right)B_w^T \\ &\stackrel{(c)}{=} \Psi_A(P_x^L(k)) + \Psi_B(P_x^L(k)), \end{aligned} \tag{13}$$

where (a) follows from the definition of Ψ_A and Ψ_B, (b) follows from (11), and (c) follows from (7). Since $P_x^M(0) = P_x^L(0) = P_o$, (12) and (13) imply $P_x^L(k) = P_x^M(k) = P_x(k)$ for every $k \in \mathbb{N}_0$. On the other hand, (8a) and (3a) imply that, $\forall k, \tau \in \mathbb{N}_0$, $R_x^L(k+\tau, k) = A_p^\tau P_x^L(k)$ and $R_x^M(k+\tau, k) = A_p^\tau P_x^M(k)$. Thus, since $P_x^L(k) = P_x^M(k) = P_x(k)$, our claim follows.
(2) Immediate from (3), (7), (8), (11), and the proof of Part (1) above. □

Lemma 8 states that, provided q satisfies Assumption 7, the second order moments of the NCS of Fig. 2(a) can be calculated by considering the simpler LTI system of Fig. 2(b).

The next result relates the MSS of the MJLS of Fig. 2(a) with the internal stability of the LTI system of Fig. 2(b):

Theorem 9 *Consider the MJLS of Fig. 2(a) and the LTI system of Fig. 2(b), where N has the realization given in (2). If Assumptions 1 and 7 hold, then the MJLS is MSS if and only if the LTI system is internally stable and*

$$1 > p(1-p)\left|\left|T_{qv}\right|\right|_2^2, \tag{14}$$

where T_{qv} is the closed loop transfer function from q to v in Fig. 2(b).

PROOF.

- ($\Rightarrow$) Note that $\mu_x^M(k) = A_p^k \mu_o$. Hence, if the MJLS is MSS, then Part 1 of Lemma 8 implies that the LTI system is internally stable . Also, Corollary 5 and the definition of limit guarantees that $R_v(\tau) \triangleq \lim_{k\to\infty} R_v(k+\tau, k)$ exists for $\tau \in \mathbb{N}_0$, and also that the stationary variance of v, $\sigma_v^2 = R_v(0)$, is finite, non-negative and unique. Hence,

$$\sigma_q^2 \triangleq \lim_{k\to\infty} P_q(k) = p(1-p)\sigma_v^2 \tag{15}$$

  is also finite, non-negative and unique.

  On the other hand, Fig. 2(b) implies that $v = T_{dv}d + T_{qv}q$ (here, T_{dv} refers to the transfer function between d and v in Fig. 2(b)). Thus, the facts in the previous paragraph allow one to write

$$\sigma_v^2 = \left|\left|T_{dv}\Omega_d\right|\right|_2^2 + \sigma_q^2 \left|\left|T_{qv}\right|\right|_2^2 = \left|\left|T_{dv}\Omega_d\right|\right|_2^2 + p(1-p)\sigma_v^2 \left|\left|T_{qv}\right|\right|_2^2, \tag{16}$$

  where we have used (15). Now, since (16) admits a unique solution for σ_v^2 if and only if (14) is satisfied, our claim follows.
- ($\Leftarrow$) This part of the proof follows by mimicking the proof of Theorem 3.2 in [16]. $\square$

Theorem 9 characterizes the MSS of the NCS of interest (equivalently, of the MJLS of Fig. 2(a)) in terms of the internal stability of the auxiliary LTI system of Fig. 2(b), and an inequality constraint. As shown in the proof of Theorem 9, the internal stability of the auxiliary LTI system is easily seen to be equivalent to the internal stability of the system $S_{\mathcal{E}}$ that governs the mean of the state in the NCS. Thus, the "only if" part of Theorem 9 is unsurprising (see also Proposition 3.6 in [5]). However, it is well known that, in general, the asymptotic stability of $S_{\mathcal{E}}$ is not sufficient for the associated MJLS to be MSS (see Remark 3.7 in [5]). Our result shows, for the class of systems under study, that the stability of $\mathcal{S}_{\mathcal{E}}$, and the additional inequality in (14), guarantee the MSS of the MJLS.

Corollary 10 *Consider the setup and assumptions of Theorem 9. Denote by T_{xy} the closed loop transfer function from x to y in the feedback loop of Fig. 2(b).*

(1) If the LTI system is internally stable and (14) *holds, then the stationary PSD of z in Fig. 2(b) is given by*

$$S_z^L = T_{dz}S_dT_{dz}^H + \sigma_q^2 T_{qz}T_{qz}^H, \tag{17}$$

where $\sigma_q^2 \in \mathbb{R}_0^+$ is the stationary variance of q, and satisfies

$$\sigma_q^2 = \frac{p(1-p)\left|\left|T_{dv}\Omega_d\right|\right|_2^2}{1 - p(1-p)\left|\left|T_{qv}\right|\right|_2^2}. \tag{18}$$

(2) If the MJLS is MSS, then $S_z^M = S_z^L$.

PROOF. Since Lemma 8 and Corollary 5 hold, it suffices to prove Part 1. From Theorem 9 and its proof, we have that (16) and (15) hold. Thus, σ_q^2 exists, is finite and satisfies (18). Also, from Fig. 2(b), we have that $z = T_{dz}d + T_{qz}q$. Since the LTI is internally stable and σ_q^2 exists, (17) follows. $\square$

Provided MSS holds, Corollary 10 gives a closed form expression for the stationary PSD of z in the NCS under study.

We conclude from Lemma 8, Theorem 9 and Corollary 10 that both instantaneous and stationary second-order and MSS-related properties of the NCS under study (equivalently, of the MJLS of Fig. 2(a)) can be studied by means of the simpler LTI system of Fig. 2(b), where the unreliable channel has been replaced by an additive white noise channel, having gain p and a fixed instantaneous SNR constraint. This key insight will be exploited in Section 5 to characterize the solution of Problem 6.

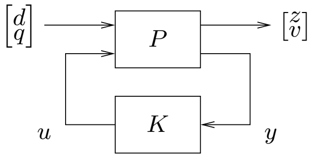

Fig. 3. Rewriting of the NCS of Fig. 1 using the equivalent additive channel noise model.

A particular instance of Theorems 9 and Corollary 10 has been previously reported in [15, 16]. However, the results in [15, 16] were presented only for a specific NCS architecture. To the best of the authors knowledge, both the instantaneous second order equivalence revealed by Lemma 8, and the connections between unreliable channels and SNR constrained additive noise channels, are new results. [3]

5 Design of Controllers implemented over Unreliable Channels

This section presents the main result of this paper. Namely, we show that Problem 6 is essentially equivalent to a standard quadratic optimal control problem.

Consider the NCS of Fig. 1. Using the results of Section 4, we rewrite the NCS as shown in Fig. 3, where P is partitioned such that

$$\begin{bmatrix} z \\ \hline v \\ \hline y \end{bmatrix} = \left[\begin{array}{c|c} P_{11}^z & P_{12}^z \\ \hline P_{11}^v & P_{12}^v \\ \hline P_{21} & P_{22} \end{array}\right] \begin{bmatrix} d \\ \underline{q} \\ u \end{bmatrix}. \tag{19}$$

We note that, provided Assumption 2(b) holds, P_{22} is strictly proper. The realization that P_{22} inherits when the system of Fig. 1 is rewritten as in Fig. 3, will be referred to as the inherited realization of P_{22}.

Lemma 11 *Consider the NCS of Fig. 1, suppose that Assumptions 1 and 2 hold, and also that the inherited realization of P_{22} is stabilizable and detectable. Consider a doubly coprime factorization of P_{22} over $\mathcal{RH}_\infty$, i.e., consider X_i, Y_i, X_d, Y_d, N_i, D_i, N_d, $D_d \in \mathcal{RH}_\infty$, with X_i, X_d, D_i, D_d biproper, such that $P_{22} = N_d D_d^{-1} = D_i^{-1} N_i$ and*

$$\begin{bmatrix} X_i & -Y_i \\ -N_i & D_i \end{bmatrix} \begin{bmatrix} D_d & Y_d \\ N_d & X_d \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{20}$$

(1) The controller K belongs to $\mathcal{S}_p$ (see definition in Problem 6) if and only if there exists $Q \in \mathcal{RH}_\infty$ such that

$$K = (X_i - QN_i)^{-1}(Y_i - QD_i), \tag{21}$$

and

$$p(1-p)\,||A_o - B_o Q C_o||_2^2 < 1, \tag{22}$$

where $A_o \triangleq (P_{11}^v + P_{12}^v D_d Y_i P_{21})\eta$, $B_o \triangleq P_{12}^v D_d$, and $C_o \triangleq D_i P_{21}\eta$ are transfer functions in $\mathcal{RH}_\infty$, and $\eta \triangleq [0 \ \cdots \ 0 \ 1]^T$.

(2) The set $\mathbb{P}$ of all successful transmission probabilities $p \in (0,1)$ that make $\mathcal{S}_p$ non-empty (hence ensuring the feasibility of Problem 6) is given by

$$\mathbb{P} = \left\{ p \in (0,1) : p(1-p) \inf_{Q \in \mathcal{RH}_\infty} ||A_o - B_o Q C_o||_2^2 < 1 \right\}, \tag{23}$$

with A_o, B_o, C_o as above.

[3] Preliminary stationary versions of these results were reported by us in [21, 22].

PROOF. Our claims follow upon using Theorem 9, the well-known Youla-Kucera parameterization of all stabilizing controllers for LTI plants (see, e.g., [26]), (19), and the definition of $\mathbb{P}$. □

We next use Lemma 11 to prove the main result of this section. To that end, we introduce a simplifying assumption:

Assumption 12 *If Problem 6 is feasible, then the optimal controller K is such that the closed loop transfer function from d to v in Fig. 3 satisfies $T_{dv}\Omega_d \not\equiv 0$.* ■

Assumption 12 is reasonable. Indeed, if it did not hold, then optimal performance would be achieved without sending any information about d over the communication channel. Clearly, that case is uninteresting when studying NCSs.

Theorem 13 *Consider the NCS of Fig. 1 and suppose that Assumptions 1 and 2 hold. If the inherited realization of P_{22} is stabilizable and detectable, $p \in \mathbb{P}$, and Assumption 12 holds, then (recall Problem 6)*

$$\left[\sigma_z^2\right]_p = \inf_{\sigma_q^2 \in \mathbb{R}_0^+} \inf_{\substack{Q \in \mathcal{RH}_\infty \\ R_{\sigma_q^2}(Q) \leq \sigma_q^2 \frac{1}{p(1-p)}}} J_{\sigma_q^2}(Q), \tag{24}$$

where, for any given $\sigma_q^2 \in \mathbb{R}_0^+$,

$$J_{\sigma_q^2}(Q) \triangleq \left|\left|\left(A_1 - B_1 Q C_1\right)\Omega\right|\right|_2^2, \quad R_{\sigma_q^2}(Q) \triangleq \left|\left|\left(A_2 - B_2 Q C_2\right)\Omega\right|\right|_2^2, \tag{25}$$

with $A_i, B_i, C_i \in \mathcal{RH}_\infty$,

$$A_1 \triangleq P_{11}^z + P_{12}^z D_d Y_i P_{21}, \quad B_1 \triangleq P_{12}^z D_d, \quad C_1 \triangleq D_i P_{21}, \tag{26}$$

$$A_2 \triangleq P_{11}^v + P_{12}^v D_d Y_i P_{21}, \quad B_2 \triangleq P_{12}^v D_d, \quad C_2 \triangleq D_i P_{21}, \tag{27}$$

and $\Omega \triangleq \operatorname{diag}\{\Omega_d, \sigma_q\}$.

PROOF. For any $K \in \mathcal{S}_p$, Theorem 9, and its proof, imply that the stationary variances of both v and q exist, and are related via $\sigma_q^2 = p(1-p)\sigma_v^2$. Thus, Part 1 of Corollary 10 with $z = v$, allows one to write

$$\sigma_v^2 = \frac{\sigma_q^2}{p(1-p)} = \left|\left|T_{dv}\Omega_d\right|\right|_2^2 + \sigma_q^2 \left|\left|T_{qv}\right|\right|_2^2 = \left|\left|\left[T_{dv} \;\; T_{qv}\right]\Omega\right|\right|_2^2. \tag{28}$$

Corollary 10 also implies that the stationary variance of z satisfies

$$\sigma_z^2 = \left|\left|T_{dz}\Omega_d\right|\right|_2^2 + \sigma_q^2 \left|\left|T_{qz}\right|\right|_2^2 = \left|\left|\left[T_{dz} \;\; T_{qz}\right]\Omega\right|\right|_2^2, \tag{29}$$

for any $K \in \mathcal{S}_p$. Given (21) and (19), it follows from Fig. 3 that (29) can be rewritten as $\sigma_z^2 = J_{\sigma_q^2}(Q)$, and (28) as $\sigma_v^2 = R_{\sigma_q^2}(Q)$. On the other hand, (18) implies that (14) (equivalently (22)) holds if and only if $\sigma_q^2 \in \mathbb{R}_0^+$. We thus conclude that

$$\left[\sigma_z^2\right]_p \triangleq \inf_{\substack{Q \in \mathcal{RH}_\infty,\, \sigma_q^2 \in \mathbb{R}_0^+ \\ R_{\sigma_q^2}(Q) = \sigma_q^2 \frac{1}{p(1-p)}}} J_{\sigma_q^2}(Q). \tag{30}$$

To complete the proof, we next show that the inequality constraint in (24) is active at the optimum. We rewrite (28) as

$$\frac{\sigma_v^2}{\sigma_q^2} = \frac{\left|\left|T_{dv}\Omega_d\right|\right|_2^2}{\sigma_q^2} + \left|\left|T_{qv}\right|\right|_2^2 = \frac{1}{p(1-p)}. \tag{31}$$

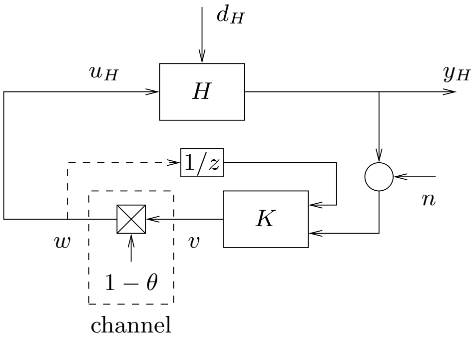

Fig. 4. SISO plant controlled over an unreliable channel.

Since $T_{dv}\Omega_d \not\equiv 0$ at the optimum, (31) shows that, if the inequality constraint in (24) is not active at the optimum, then one can always reduce the value of σ_q^2 so as to make the inequality constraint active. Thus, if $T_{qz} \equiv 0$ at the optimum, then one can always pick σ_q^2 so as to make the inequality constraint in (24) active, without compromising optimality. If, on the other hand, $T_{qz} \not\equiv 0$ at the optimum, then reducing σ_q^2 implies a decrease in σ_z^2 (see (29)), which contradicts optimality. This completes the proof. □

The characterization of the solution of Problem 6 provided by Theorem 13 is not explicit. However, since both $J_{\sigma_q^2}$ and $R_{\sigma_q^2}$ are convex functions of Q, the inner problem in (24) is a standard convex quadratic optimal control problem that can be solved using standard tools including LMIs (see, e.g., [2,3]). The outer problem in (24) is a line search and poses no numerical difficulties. A detailed study of the optimization problem in (24) can be found in [23].

It is worth noting that Theorem 13 is a direct consequence of the equivalence between unreliable channels and SNR constrained additive noise channels established in Section 4.

6 Control of SISO Plants over Unreliable Channels

In this section we illustrate our results by focusing on the control of a SISO LTI plant H modelled by

$$\begin{bmatrix} x_H(k+1) \\ y_H(k) \end{bmatrix} = \begin{bmatrix} A_H & B_{H,u} & B_{H,d} \\ C_H & 0 & D_{H,d} \end{bmatrix} \begin{bmatrix} x_H(k) \\ u_H(k) \\ d_H(k) \end{bmatrix}, \quad k \in \mathbb{N}_0, \quad x_H(0) = x_{H,o}, \tag{32}$$

where x_H is the n_{x_H}-dimensional state, u_H is the scalar control input, d_H is an m-dimensional disturbance signal, and y_H is a scalar output that can be measured. Our aim is to control H over a scalar erasure channel. We restrict ourselves to LTI controllers and utilize the architecture shown in Fig. 4. In that figure, n models measurement noise and K is the LTI controller.

The dashed line in Fig. 4 suggests that one-step-delayed feedback around the channel may be exploited by the controller K. If such feedback is available, then it is implicitly assumed that the communication protocol provides perfect packet acknowledgements (TCP-like protocols; see, e.g., [12,19]). If no acknowledgements are available (UDP-like protocols), then K can only use $n + y_H$ to construct the channel input v.

The next result presents conditions for the existence of controllers K that render the NCS of Fig. 4 MSS:

Theorem 14 *Consider the NCS of Fig. 4 and assume that H is unstable, that $(A_H, B_{H,u}, C_H, 0)$ is both detectable and stabilizable, that the initial states of H and K, say $x_{H,o}$ and $x_{K,o}$, are second order random variables, and that $[d_H^T \;\; n]^T$ is a zero mean second order wss sequence uncorrelated with $[x_{H,o}^T \;\; x_{K,o}^T]^T$. If the dropout process θ is independent of $(x_{H,o}, x_{K,o}, d_H, n)$ and satisfies Assumption 1(b), then:*

(1) *If one-step-delayed feedback around the channel is available (TCP-like protocols), then there exists a proper LTI K that achieves MSS if and only if*

$$p > p_{\inf} \triangleq 1 - \frac{1}{\prod_{i=1}^{n_p} |p_i|^2}, \tag{33}$$

where $p_1, \cdots, p_{n_p}$ are the unstable poles of H (i.e., the eigenvalues of A_H in $\{z \in \mathbb{C} : |z| \geq 1\}$).

(2) *If no feedback around the channel is available (UDP-like protocols), then there exists a proper LTI K that achieves MSS if and only if*

$$p > 1 - \frac{1}{\left(\prod_{i=1}^{n_p} |p_i|^2\right) + \Delta}, \tag{34}$$

where $p_1, \cdots, p_{n_p}$ are the unstable poles of H, and $\Delta \triangleq \eta + \delta \geq 0$, with δ and η defined as in Equation (34) in [4] with $G_d = C_H(zI - A_H)^{-1}B_{H,u}$. Moreover, $\Delta = 0$ if and only if the plant has relative degree one and has no zeros in $\{z \in \mathbb{C} : |z| > 1\}$.

PROOF. Our assumptions together with the structure assumed for H (see (32)), imply that the scheme of Fig. 4, when rewritten as in the general form of Fig. 1, satisfies Assumptions 1(a) and 2, and also that the assumptions of Lemma 11 are satisfied. Define $G \triangleq C_H(zI - A_H)^{-1}B_{H,u}$.

(1) We rewrite the scheme of Fig. 4 in the general form of Fig. 1. Given our assumptions, Lemma 11 is applicable and hence suffices to characterize the set $\mathbb{P}$ in (23) to prove our claim.

If feedback around the channel is available, then inspection of Fig. 4 reveals that $P_{22} = p[G \;\; z^{-1}]^T$. Assume that $N, D \in \mathcal{RH}_\infty$ form a coprime factorization of G over $\mathcal{RH}_\infty$. A coprime factorization of P_{22} is thus given by $P_{22} = N_d D_d^{-1} = D_i^{-1} N_i$, with $N_d \triangleq [pN \;\; pz^{-1}D]^T$, $D_d \triangleq D$, $N_i \triangleq [pN \;\; pz^{-1}]$, $D_i \triangleq \operatorname{diag}\{D, 1\}$, and A_o, B_o, C_o are given by $A_o = D_d Y_i [G \;\; z^{-1}]^T = p^{-1} Y_i N_d$, $B_o = D_d = D$, and $C_o = D_i [G \;\; z^{-1}]^T = p^{-1} N_i$.

Using the proof of Theorem 17 in [23], we conclude that

$$p^2 \left|\left|A_o - B_o Q C_o\right|\right|_2^2 \geq \left(\prod_{i=1}^{n_p} |p_i|^2\right) - 1, \tag{35}$$

where the gap between both sides of (35) can be made arbitrarily small with some $Q \in \mathcal{RH}_\infty$. It is now immediate to conclude that $\mathbb{P} = \left\{p \in (0,1) : \frac{p(1-p)}{p^2}\left(\prod_{i=1}^{n_p} |p_i|^2 - 1\right) < 1\right\}$, from where Part 1 readily follows.

(2) This part follows by proceeding as above and exploiting Theorem III.2 in [4]. □

Remark 15 *If H is stable, then $K = 0$ achieves MSS in the NCS of Fig. 4 for any p.* ■

Theorem 14 gives an explicit characterization of the minimal successful transmission probability, p, that allows one to design a controller K that guarantees MSS in the NCS of Fig. 4. As expected, the degree of instability of the plant (as measured by the product of its unstable poles) plays a key role in the channel reliability required to achieve MSS. When TCP-like protocols are used, unstable poles are the only source of limitation on the admissible values for p. However, when UDP-like protocols are employed, plant non-minimum phase zeros and relative degree also play a role and, in general, more stringent conditions on p arise. (The requirements on p are the same in both situations if and only if the plant has relative degree one and no zeros in $\{z \in \mathbb{C} : |z| > 1\}$.)

In the TCP-like protocol case, our results are consistent with the results in [19] (see also [12]). However, our results hold when LTI controllers are employed, whereas the results in [19] consider time varying control schemes. Stated another way, for any given scalar analog erasure channel and a TCP-like communication protocol, the class of SISO plant models for which the time varying schemes of [19] achieve MSS, is the same class for which our proposal does so. This implies that time varying control architectures provide no aditional advantages over LTI ones, when the plant is LTI, SISO, and mean square stabilization is the only control objective.

In the UDP-like protocol case, the results of Theorem 6.1 in [19] [4] give explicit necessary conditions on the dropout probabilities that guarantee MSS for the case of plant models that have a square and invertible state-to-output matrix (the "C" matrix). That condition turns out to be also sufficient in the case of systems that, in addition, have a square and invertible input-to-state matrix ("B" matrix; see [12]). In the SISO case, having both a square input-to-state and state-to-output matrix is tantamount to having a one-dimensional (i.e., scalar) plant model. Our results are not only consistent with the results mentioned above, but they also provide both necessary and sufficient conditions which are valid for any SISO LTI plant, controlled using the architecture of Fig. 4.

Theorem 14 also extends the results of Section 8 in [6]. In that work, the author shows that (33) is necessary and sufficient to be able to find a one-degree of freedom LTI controller (that does not use packet acknowledgements), so as to achieve MSS in an NCS built around a single input plant whose state can be noiselessly measured. Part 2 of Theorem 14 shows that, in the (noisy) output feedback case, more than the unstable plant poles limit the channel reliability required to achieve MSS. Plant non-minimum phase zeros and relative degree also play a role. (This fact was loosely mentioned in Section V.B in [7], but no closed form expression quantifying the effect of non-minimum phase zeros or relative degree was presented.)

If, in Fig. 4, one is interested in more than the stability issue, then, irrespective of whether one assumes feedback around the channel or not, it suffices to use Theorem 13 to obtain an optimization problem that can be tackled using standard tools, and whose solution yields an LTI controller K that optimizes performance.

The performance achieved by the controller that our methodology suggests is only optimal within the class of control architectures under consideration, i.e., when the architecture of Fig. 4 is used and K is LTI. Obviously, better performance may be attainable if one considers time varying controllers or more complex architectures as proposed in, e.g., [19, 20]. Indeed, it has been shown in [19] that optimal control policies for control problems that involve unreliable channels are, in general, time varying and, in the case of employing UDP-like protocols, non-linear. Our approach, although suboptimal, allows one to achieve performance levels that may be acceptable in practice.

The work [24] also provides a characterization of optimal linear controllers for NCSs that are subject to data loss. Our results, as opposed to those in [24], reveal the fact that a solution to Problem 6 can be found using only well-known optimization methods. [5]

Remark 16 *It is possible to use our framework to study LTI control architectures that are more complex than that of Fig. 4. For example, one can include an LTI system M between the output of the channel w and the input of the plant u_H (see Fig. 4). Interestingly, Theorem 14 still applies in that case (see details in Chapter 7 of [21]). However, a difficulty arises when attempting to optimally designing this new control architecture. Since the communication constraint precludes the use of the measured plant output at the receiving end, rewriting the resulting feedback system in the standard form of Fig. 3 results in a controller K with a sparsity constraint (see, e.g., [18]). Thus, the optimal design problem becomes a non-convex problem [25]. To circumvent this complication, one can proceed in an iterative fashion: first, design K for a given M; then, design M for the previously chosen K, etc. Each of these steps yields a problem that fits into the setup of Theorem 13.* ■

Example 17 *Consider a plant H such that*

$$y_H = \frac{-0.25(z-2)}{(z^2+z+1.5)}(u_H + d_H). \tag{36}$$

Assume that $n = 0$ and that d_H is a zero mean white noise sequence with unitary variance.

If packet acknowledgements are exploited by K, then $p > 0.5556$ is necessary and sufficient to be able to find a stabilizing K. If no packet acknowledgements are exploited, then the condition becomes $p > 0.6032$. In both cases, we have computed the best performance achievable using the architecture of Fig. 4 for several values of p. The results are presented in Fig. 5(a). As expected, the performance becomes increasingly worse as p approaches the minimal p compatible with MSS in each of the cases. It is also unsurprising that, as $p \to 1$, we recover the best possible non-networked performance irrespective of whether acknowledgements are exploited by K or not. Fig. 5(b) shows the

[4] Note that equation (40) in Theorem 6.1 in [19] contains a typo that is corrected in the corresponding proof.

[5] We note, however, that the work in [24] focuses on architectures with two unreliable channels (one for the controller-to-actuator link, and one for the sensor-to-controller link).

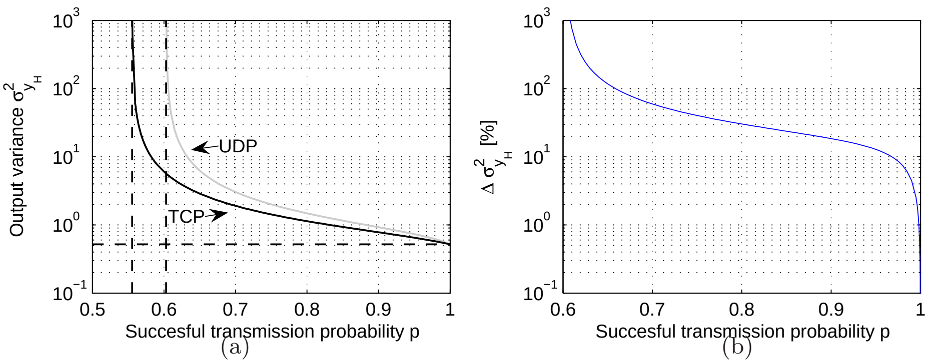

Fig. 5. (a) Best performance achievable with the architecture of Fig. 4 when both TCP-like and UDP-like protocols are employed, and (b) relative performance deterioration when K does not use packet acknowledgements, with respect to the situation where acknowledgements are used.

relative performance deterioration $\Delta\sigma^2_{y_H}$ (in %) when K does not use packet acknowledgements, with respect to the situation where such acknowledgements are used. The performance gains arising form the use of acknowledgements are significative over a wide range of values for p. ■

7 Conclusions

This paper has studied LTI control architectures that comprise a scalar unreliable channel in the feedback path. We have shown that, when data-dropouts are governed by a Bernoulli process, the unreliable channel is equivalent to an additive white noise channel with an instantaneous signal-to-noise ratio constraint. This key insight has enabled us to restate an optimal control problem over unreliable channels, in terms of a standard quadratic problem.

To illustrate our results, we have studied the control of SISO plants. For this situation, we were able to establish closed form necessary and sufficient conditions on the minimal successful transmission probability that allows one to design LTI controllers that achieve MSS. This was possible even when UDP-like communication protocols are employed, thus extending previous results reported in [6, 7, 12, 19].

Future work should focus on situations where multiple channels are used, and on the MIMO case.

8 Acknowledgments

The authors would like to thank Prof. Graham C. Goodwin and Dr. Daniel E. Quevedo (The University of Newcastle, Australia) for helpful discussions and suggestions. E.I. Silva acknowledges support from CONICYT through grant FONDECYT 3100024.

A Appendix

Fact 18 *Consider $X_1, X_2 \in \mathbb{R}^{n_x \times m_x}$, $Y_1, Y_2 \in \mathbb{R}^{n_y \times m_y}$, and $p, d_r \in \mathbb{R}$. Define $X_p \triangleq X_1 + pX_2$, $Y_p \triangleq Y_1 + pY_2$, $X(d_r) \triangleq X_1 + (1-d_r)X_2$, $Y(d_r) \triangleq Y_1 + (1-d_r)Y_2$. Then,*

$$pX(0)PY(0)^T + (1-p)X(1)PY(1)^T = X_p P Y_p^T + p(1-p)X_2 P Y_2^T \tag{A.1}$$

for any $P \in \mathbb{R}^{m_x \times m_y}$.

PROOF. Immediate upon expanding both sides of (A.1). □

References

[1] P. Antsaklis and J. Baillieul. Special issue on technology of networked control systems. *Proceedings of the IEEE*, 95(1):5–8, 2007.

[2] S. Boyd and C. Barratt. *Linear Controller Design: Limits of Performance.* Prentice Hall, New Jersey, 1991.

[3] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM, 1994.

[4] J.H. Braslavsky, R.H. Middleton, and J.S. Freudenberg. Feedback stabilization over signal-to-noise ratio constrained channels. *IEEE Transactions on Automatic Control*, 52(8):1391–1403, 2007.

[5] O.L.V. Costa, M.D. Fragoso, and R.P. Marques. *Discrete Time Markov Jump Linear Systems.* Springer, 2005.

[6] N. Elia. Remote stabilization over fading channels. *Systems & Control Letters*, 54(3):237–249, 2005.

[7] N. Elia and J.N. Eisenbeis. Limitations of linear remote control over packet drop networks. In *Proc. of the 43rd IEEE Conference on Decision and Control*, Atlantis, Bahamas, December 2004.

[8] V. Gupta, B. Hassibi, and R.M. Murray. Optimal LQG control across packet-dropping links. *Systems & Control Letters*, 56(6):439–446, 2007.

[9] C.N. Hadjicostis and R. Touri. Feedback control utilizing packet dropping network links. In *Proceedings of the 41st IEEE Conference on Decision and Control*, Las Vegas, USA, 2002.

[10] A. Hassibi, S. Boyd, and J.P. How. Control of asynchronous dynamical systems with rate constraints on events. In *Proceedings of the 38th IEEE Conference on Decision and Control*, Phoenix, USA, 1999.

[11] J.P. Hespanha, P. Naghshtabrizi, and Y. Xu. A survey of recent results in networked control systems. *Proceedings of the IEEE*, 95(1):138–162, January 2007.

[12] O.C. Imer, S. Yüksel, and T. Başar. Optimal control of LTI systems over unreliable communication links. *Automatica*, 42:1429 – 1439, 2006.

[13] H. Ishii and S. Hara. A subband coding approach to networked control. In *Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems*, Kyoto, Japan, 2006.

[14] F. Kozin. A survey of stability of stochastic systems. *Automatica*, 5:95–112, 1969.

[15] Q. Ling. *Stability and performance of control systems with limited feedback information.* PhD thesis, University of Notre Dame, Notre Dame, USA, 2005.

[16] Q. Ling and M.D. Lemmon. Power spectral analysis of networked control systems with data dropouts. *IEEE Transactions on Automatic Control*, 49(6):955–960, June 2004.

[17] J. Nilsson. *Real-time Control Systems with Delays.* PhD thesis, Lund Institute of Technology, Sweden, 1998.

[18] M. Rotkowitz and S. Lall. A characterization of convex problems in decentralized control. *IEEE Transactions on Automatic Control*, 51(2):274–286, 2006.

[19] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. Sastry. Foundations of control and estimation over lossy networks. *Proceedings of the IEEE*, 95(1):163 – 187, January 2007.

[20] P. Seiler and R. Sengupta. An $\mathcal{H}_\infty$ approach to networked control. *IEEE Transactions on Automatic Control*, 50(3), 2005.

[21] E.I. Silva. *A Unified Framework for the Analysis and Design of Networked Control Systems.* PhD thesis, School of Electrical Eng. and Comp. Sci., The University of Newcastle, Australia, 2009.

[22] E.I. Silva, G.C. Goodwin, and D.E. Quevedo. On the design of control systems over unreliable channels. In *Proceedings of the European Control Conference*, Budapest, Hungary, 2009.

[23] E.I. Silva, G.C. Goodwin, and D.E. Quevedo. Control system design subject to SNR constraints. *Automatica*, 46(2):428–436, 2010.

[24] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, and S.S. Sastry. Optimal linear LQG control over lossy networks without packet acknowlegements. *Asian Journal of Control*, 10(1), January 2008.

[25] D. Sourlas and V. Manousiousthakis. Best achievable decentralized performance. *IEEE Transactions on Automatic Control*, 40(11):1858–1871, November 1995.

[26] M. Vidyasagar. *Control Systems Synthesis: A Factorization Approach.* MIT Press, Cambridge, USA, 1985.

[27] W. Zhang, M. Branicky, and S. Phillips. Stability of networked control systems. *IEEE Control Systems Magazine*, 21:84–99, 2001.